NOTE ON MMAT 5010: LINEAR ANALYSIS (2020-21 2ND TERM)

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1. Lecture 1

Throughout this note, we always denote $\mathbb{K}$ by the real field $\mathbb{R}$ or the complex field $\mathbb{C}$. Let $\mathbb{N}$ be the set of all natural numbers. Also, we write a sequence of numbers as a function $x : \{1, 2, ...\} \to \mathbb{K}$ or $x_i := x(i)$ for $i = 1, 2....$

Definition 1.1. *Let X be a vector space over the field $\mathbb{K}$. A function $\|\cdot\| : X \to \mathbb{R}$ is called a norm on X if it satisfies the following conditions.*

(i) $\|x\| \geq 0$ for all $x \in X$ and $\|x\| = 0$ if and only if $x = 0$.
(ii) $\|\alpha x\| = |\alpha|\|x\|$ for all $\alpha \in \mathbb{K}$ and $x \in X$.
(iii) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$.

In this case, the pair $(X, \|\cdot\|)$ is called a normed space.

Remark 1.2. *Recall that a* metric space *is a non-empty set Z together with a function, (*called a metric*), $d : Z \times Z \to \mathbb{R}$ that satisfies the following conditions:*

(i) $d(x, y) \geq 0$ for all $x, y \in Z$; and $d(x, y) = 0$ if and only if $x = y$.
(ii) $d(x, y) = d(y, x)$ for all $x, y \in Z$.
(iii) $d(x, y) \leq d(x, z) + d(z, y)$ for all x, y and z in Z.

For a normed space $(X, \|\cdot\|)$, if we define $d(x, y) := \|x - y\|$ for $x, y \in X$, then X becomes a metric space under the metric d.

The following examples are important classes in the study of functional analysis.

Example 1.3. *Consider $X = \mathbb{K}^n$. Put*

$$\|x\|_p := \big(\sum_{i=1}^{n} |x_i|^p\big)^{1/p} \quad \textit{and} \quad \|x\|_\infty := \max_{i=1,...,n} |x_i|$$

for $1 \leq p < \infty$ and $x = (x_1, ..., x_n) \in \mathbb{K}^n$.
Then $\|\cdot\|_p$ (called the usual norm as p=2) and $\|\cdot\|_\infty$ (called the sup*-norm) all are norms on $\mathbb{K}^n$.*

Example 1.4. *Put*

$$c_0 := \{(x(i)) : x(i) \in \mathbb{K},\ \lim |x(i)| = 0\} \textit{(called the null sequnce space)}$$

and

$$\ell^\infty := \{(x(i)) : x(i) \in \mathbb{K},\ \sup_i |x(i)| < \infty\}.$$

Then c_0 is a subspace of ℓ^∞. The sup*-norm $\|\cdot\|_\infty$ on ℓ^∞ is defined by*

$$\|x\|_\infty := \sup_i |x(i)|$$

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for $x \in \ell^\infty$. Let

$$c_{00} := \{(x(i)) : \text{ there are only finitly many } x(i)\text{'s are non-zero}\}.$$

Also, c_{00} is endowed with the sup-*norm defined above and is called the finite sequence space.*

Example 1.5. *For $1 \le p < \infty$, put*

$$\ell^p := \{(x(i)) : x(i) \in \mathbb{K}, \sum_{i=1}^{\infty} |x(i)|^p < \infty\}.$$

Also, ℓ^p is equipped with the norm

$$\|x\|_p := (\sum_{i=1}^{\infty} |x(i)|^p)^{\frac{1}{p}}$$

for $x \in \ell^p$. Then $\|\cdot\|_p$ is a norm on ℓ^p (see [2, Section 9.1]*).*

Example 1.6. *Let $C^b(\mathbb{R})$ be the space of all bounded continuous $\mathbb{R}$-valued functions f on $\mathbb{R}$. Now $C^b(\mathbb{R})$ is endowed with the* sup-*norm, that is,*

$$\|f\|_\infty = \sup_{x \in \mathbb{R}} |f(x)|$$

for every $f \in C^b(\mathbb{R})$. Then $\|\cdot\|_\infty$ is a norm on $C^b(\mathbb{R})$.
Also, we consider the following subspaces of $C^b(X)$.
Let $C_0(\mathbb{R})$ $\big($resp. $C_c(\mathbb{R})\big)$ be the space of all continuous $\mathbb{R}$-valued functions f on $\mathbb{R}$ which vanish at infinity (resp. have compact supports), that is, for every $\varepsilon > 0$, there is a $K > 0$ such that $|f(x)| < \varepsilon$ (resp. $f(x) \equiv 0$) for all $|x| > K$.
It is clear that we have $C_c(\mathbb{R}) \subseteq C_0(\mathbb{R}) \subseteq C^b(\mathbb{R})$.
Now $C_0(\mathbb{R})$ and $C_c(\mathbb{R})$ are endowed with the sup-*norm $\|\cdot\|_\infty$.*

From now on, we always let X be a normed sapce.

Definition 1.7. *We say that a sequence (x_n) in X converges to an element $a \in X$ if* $\lim \|x_n - a\| = 0$*, that is, for any $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that $\|x_n - a\| < \varepsilon$ for all $n \ge N$.*
In this case, (x_n) is said to be convergent and a is called a limit of the sequence (x_n).

Remark 1.8.
(i) *If (x_n) is a convergence sequence in X, then its limit is unique. In fact, if a and b both are the limits of (x_n), then we have $\|a-b\| \le \|a-x_n\|+\|x_n-b\| \to 0$. So, $\|a-b\| = 0$ which implies that $a = b$.*

We write $\lim x_n$ for the limit of (x_n) provided the limit exists.

(ii) *The definition of a convergent sequence (x_n) depends on the underling space where the sequence (x_n) sits in. For example, for each $n = 1, 2...$, let $x_n(i) := 1/i$ as $1 \le i \le n$ and $x_n(i) = 0$ as $i > n$. Then (x_n) is a convergent sequence in ℓ^∞ but it is not convergent in c_{00}.*

The following is one of the basic properties of a normed space. The proof is directly shown by the triangle inequality and a simple fact that every convergent sequence (x_n) must be *bounded*, i.e., there is a positive number M such that $\|x_n\| \le M$ for all $n = 1, 2, ...$

Proposition 1.9. *The addition $+ : (x, y) \in X \times X \mapsto x + y \in X$ and the scalar multiplication $\bullet : (\lambda, x) \in \mathbb{K} \times X \mapsto \lambda x \in X$ both are continuous maps. More precisely, if the convergent sequences*

$x_n \to x$ and $y_n \to y$ in X, then we have $x_n + y_n \to x + y$. Similarly, if a sequence of numbers $\lambda_n \to \lambda$ in $\mathbb{K}$, then we also have $\lambda_n x_n \to \lambda x$.

A sequence (x_n) in X is called a ***Cauchy sequence*** if for any $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that $\|x_m - x_n\| < \varepsilon$ for all $m, n \geq N$. We have the following simple observation.

Proposition 1.10. *Every convergent sequence in X is a Cauchy sequence.*

Proof. Let (x_n) be a convergent sequence with the limit a in X. Then for any $\varepsilon > 0$, there is a positive integer N such that $\|x_n - a\| < \varepsilon$ for all $n \geq N$. This implies that $\|x_m - x_n\| \leq \|x_n - a\| + \|a - x_m\| < 2\varepsilon$ for all $m, n \geq N$. Thus, (x_n) is a Cauchy sequence. $\square$

Remark 1.11. *The converse of Proposition 1.10 does not hold.*
For example, let X be the finite sequence space $(c_{00}, \|\cdot\|_\infty)$. If we consider the sequence $x_n := (1, 1/2, 1/3, ..., 1/n, 0, 0, ...) \in c_{00}$, then (x_n) is a Cauchy sequence but it is not a convergent sequence in c_{00}.
In fact, if we are given any element $a \in c_{00}$, then there exists a positive integer N such that $a(i) = 0$ for all $i \geq N$. Thus we always have $\|x_n - a\|_\infty \geq 1/N$ for all $n \geq N$ and thus, $\|x_n - a\|_\infty \nrightarrow 0$. This implies that the sequence (x_n) does not converge to any element in c_{00}.

The following notation plays an important role in mathematics.

Definition 1.12. *A normed space X is said to be a Banach space if every Cauchy sequence in X must be convergent. The space X is also said to be complete in this case.*

Example 1.13. *With the notation as above, we have the following examples of Banach spaces.*

(i) *If $\mathbb{K}^n$ is equipped with the usual norm, then $\mathbb{K}^n$ is a Banach space.*

(ii) *ℓ^∞ is a Banach space. In fact, if (x_n) is a Cauchy sequence in ℓ^∞, then for any $\varepsilon > 0$, there is $N \in \mathbb{N}$, we have*

$$|x_n(i) - x_m(i)| \leq \|x_n - x_m\|_\infty < \varepsilon$$

*for all $m, n \geq N$ and $i = 1, 2....$ Thus, if we fix $i = 1, 2, ..,$ then $(x_n(i))_{n=1}^\infty$ is a Cauchy sequence in $\mathbb{K}$. Since $\mathbb{K}$ is complete, the limit $\lim_n x_n(i)$ exists in $\mathbb{K}$ for all $i = 1, 2....$ Nor for each $i = 1, 2...,$ we put $z(i) := \lim_n x_n(i) \in \mathbb{K}$. Then we have $z \in \ell^\infty$ and $\|z - x_n\|_\infty \to 0$. So, $\lim_n x_n = z \in \ell^\infty$ (**Check !!!!**). Thus ℓ^∞ is a Banach space.*

(iii) *ℓ^p is a Banach space for $1 \leq p < \infty$. The proof is similar to the case of ℓ^∞.*

(iv) *$C[a, b]$ is a Banach space.*

(v) *Let $C_0(\mathbb{R})$ be the space of all continuous $\mathbb{R}$-valued functions f on $\mathbb{R}$ which are vanish at infinity, that is, for every $\varepsilon > 0$, there is a $M > 0$ such that $|f(x)| < \varepsilon$ for all $|x| > M$. Now $C_0(\mathbb{R})$ is endowed with the* sup*-norm, that is,*

$$\|f\|_\infty = \sup_{x \in \mathbb{R}} |f(x)|$$

for every $f \in C_0(\mathbb{R})$. Then $C_0(\mathbb{R})$ is a Banach space.

Notation 1.14. *For $r > 0$ and $x \in X$, let*

(i) *$B(x, r) := \{y \in X : \|x - y\| < r\}$ (called an open ball with the center at x of radius r) and $B^*(x, r) := \{y \in X : 0 < \|x - y\| < r\}$*

(ii) *$B(x, r) := \{y \in X : \|x - y\| \leq r\}$ (called a closed ball with the center at x of radius r).*

Put $B_X := \{x \in X : \|x\| \leq 1\}$ and $S_X := \{x \in X : \|x\| = 1\}$ the closed unit ball and the unit sphere of X respectively.

Definition 1.15. *Let A be a subset of X.*

(i) *A point $a \in A$ is called an interior point of A if there is $r > 0$ such that $B(a, r) \subseteq A$. Write $int(A)$ for the set of all interior points of A.*
(ii) *A is called an open subset of X if $int(A) = A$.*

Example 1.16. *We keep the notation as above.*

(i) *Let $\mathbb{Z}$ and $\mathbb{Q}$ denote the set of all integers and rational numbers respectively If $\mathbb{Z}$ and $\mathbb{Q}$ both are viewed as the subsets of $\mathbb{R}$, then $int(\mathbb{Z})$ and $int(\mathbb{Q})$ both are empty.*
(ii) *The open interval $(0, 1)$ is an open subset of $\mathbb{R}$ but it is not an open subset of $\mathbb{R}^2$. In fact, $int(0, 1) = (0, 1)$ if $(0, 1)$ is considered as a subset of $\mathbb{R}$ but $int(0, 1) = \emptyset$ while $(0, 1)$ is viewed as a subset of $\mathbb{R}^2$.*
(iii) *Every open ball is an open subset of X (**Check!!**).*

Definition 1.17. *Let A be a subset of X.*

(i) *A point $z \in X$ is called a limit point of A if for any $\varepsilon > 0$, there is an element $a \in A$ such that $0 < \|z - a\| < \varepsilon$, that is, $B^*(z, \varepsilon) \cap A \neq \emptyset$ for all $\varepsilon > 0$.*
Furthermore, if A contains the set of all its limit points, then A is said to be closed in X.
(ii) *The closure of A, write $\overline{A}$, is defined by*

$$\overline{A} := A \cup \{z \in X : z \text{ is a limit point of } A\}.$$

Remark 1.18. *With the notation as above:*

(i) *A set A is closed if and only if the following condition holds:*
if (x_n) is a sequence in A and is convergent in X, then $\lim x_n \in A$.
(ii) *A point $z \in \overline{A}$ if and only if $B(z, r) \cap A \neq \emptyset$ for all $r > 0$. This is also equivalent to saying that there is a sequence (x_n) in A such that $x_n \to z$. In fact, this can be shown by considering $r = \frac{1}{n}$ for $n = 1, 2....$*

Proposition 1.19. *With the notation as before, we have the following assertions.*

(i) *A is closed in X if and only if its complement $X \setminus A$ is open in X.*
(ii) *The closure $\overline{A}$ is the smallest closed subset of X containing A. The "smallest" in here means that if F is a closed subset containing A, then $\overline{A} \subseteq F$.*
Consequently, A is closed if and only if $\overline{A} = A$.

Proof. If A is empty, then the assertions (i) and (ii) both are obvious. Now assume that $A \neq \emptyset$. For part (i), let $C = X \setminus A$ and $b \in C$. Suppose that A is closed in X. If there exists an element $b \in C \setminus int(C)$, then $B(b, r) \subsetneqq C$ for all $r > 0$. This implies that $B(b, r) \cap A \neq \emptyset$ for all $r > 0$ and hence, b is a limit point of A since $b \notin A$. It contradicts to the closeness of A. So, $C = int(C)$ and thus, C is open.
For the converse of (i), assume that C is open in X. Assume that A has a limit point z but $z \notin A$. Since $z \notin A$, $z \in C = int(C)$ because C is open. Hence, we can find $r > 0$ such that $B(z, r) \subseteq C$. This gives $B(z, r) \cap A = \emptyset$. This contradicts to the assumption of z being a limit point of A. So, A must contain all of its limit points and hence, it is closed.

For part (ii), we first claim that $\overline{A}$ is closed. Let z be a limit point of $\overline{A}$. Let $r > 0$. Then there is $w \in B^*(z, r) \cap \overline{A}$. Choose $0 < r_1 < r$ small enough such that $B(w, r_1) \subseteq B^*(z, r)$. Since w is a limit point of A, we have $\emptyset \neq B^*(w, r_1) \cap A \subseteq B^*(z, r) \cap A$. So, z is a limit point of A. Thus, $z \in \overline{A}$ as required. This implies that $\overline{A}$ is closed.
It is clear that $\overline{A}$ is the smallest closed set containing A.

The last assertion follows from the minimality of the closed sets containing A immediately. The proof is finished. $\square$

Example 1.20. *Retains all notation as above. We have* $\overline{c_{00}} = c_0 \subseteq \ell^\infty$.
Consequently, c_0 *is a closed subspace of* ℓ^∞ *but* c_{00} *is not.*

Proof. We first claim that $\overline{c_{00}} \subseteq c_0$. Let $z \in \ell^\infty$. It suffices to show that if $z \in \overline{c_{00}}$, then $z \in c_0$, that is, $\lim_{i\to\infty} z(i) = 0$. Let $\varepsilon > 0$. Then there is $x \in B(z, \varepsilon) \cap c_{00}$ and hence, we have $|x(i) - z(i)| < \varepsilon$ for all $i = 1, 2.....$ Since $x \in c_{00}$, there is $i_0 \in \mathbb{N}$ such that $x(i) = 0$ for all $i \geq i_0$. Therefore, we have $|z(i)| = |z(i) - x(i)| < \varepsilon$ for all $i \geq i_0$. So, $z \in c_0$ as desired.

For the reverse inclusion, let $w \in c_0$. It needs to show that $B(w, r) \cap c_{00} \neq \emptyset$ for all $r > 0$. Let $r > 0$. Since $w \in c_0$, there is i_0 such that $|w(i)| < r$ for all $i \geq i_0$. If we let $x(i) = w(i)$ for $1 \leq i < i_0$ and $x(i) = 0$ for $i \geq i_0$, then $x \in c_{00}$ and $\|x - w\|_\infty := \sup_{i=1,2...} |x(i) - w(i)| < r$ as required. $\square$

Proposition 1.21. *Let* Y *be a subspace of a Banach space* X*. Then* Y *is a Banach space if and only if* Y *is closed in* X*.*

Proof. For the necessary condition, we assume that Y is a Banach space. Let $z \in \overline{Y}$. Then there is a sequence (y_n) in Y such that $y_n \to z$. Since (y_n) is convergent, it is also a Cauchy sequence in Y. Then (y_n) is a convergent sequence in Y because Y is a Banach space. Therefore, $z \in Y$. This implies that $\overline{Y} = Y$ and hence, Y is closed.
For the converse statement, assume that Y is closed. Let (z_n) be a Cauchy sequence in Y. Then it is also a Cauchy sequence in X. Since X is complete, $z := \lim z_n$ exists in X. Note that $z \in Y$ because Y is closed. Thus, (z_n) is convergent in Y and Y is complete as desired. $\square$

Corollary 1.22. c_0 *is a Banach space but the finite sequence* c_{00} *is not.*

Proposition 1.23. *Let* $(X, \|\cdot\|)$ *be a normed space. Then there is a normed space* $(X_0, \|\cdot\|_0)$*, together with a linear map* $i : X \to X_0$*, satisfy the following condition.*

(i) X_0 *is a Banach space.*
(ii) The map i *is an isometry, that is,* $\|i(x)\|_0 = \|x\|$ *for all* $x \in X$*.*
(iii) the image $i(X)$ *is dense in* X_0*, that is,* $\overline{i(X)} = X_0$*.*

Moreover, such pair (X_0, i) *is unique up to isometric isomorphism in the following sense: if* $(W, \|\cdot\|_1)$ *is a Banach space and an isometry* $j : X \to W$ *is an isometry such that* $\overline{j(X)} = W$*, then there is an isometric isomorphism* ψ *from* X_0 *onto* W *such that*

$$j = \psi \circ i : X \to X_0 \to W.$$

In this case, the pair (X_0, i) *is called the completion of* X*.*

Example 1.24. *Proposition 1.23 cannot give an explicit form of the completion of a given normed space. The following examples are basically due to the uniqueness of the completion.*

(i) If X *is a Banach space, then the completion of* X *is itself.*
(ii) By Corollary 1.22, the completion of the finite sequence space c_{00} *is the null sequence space* c_0*.*
(iii) The completion of $C_c(\mathbb{R})$ *is* $C_0(\mathbb{R})$*.*

2. Lecture 2

Throughout this section, (X, d) always denotes a metric space. Let (x_n) be a sequence in X. Recall that a subsequence $(x_{n_k})_{k=1}^{\infty}$ of (x_n) means that $(n_k)_{k=1}^{\infty}$ is a sequence of positive integers satisfying $n_1 < n_2 < \cdots < n_k < n_{k+1} < \cdots$, that is, such sequence (n_k) can be viewed as a strictly increasing function $\mathbf{n} : k \in \{1, 2, ..\} \mapsto n_k \in \{1, 2, ...\}$.
In this case, note that for each positive integer N, there is $K \in \mathbb{N}$ such that $n_K \geq N$ and thus we have $n_k \geq N$ for all $k \geq K$.

Proposition 2.1. *Let (x_n) be a sequence in X. Then the following statements are equivalent.*

(i) (x_n) is convergent.
(ii) Any subsequence (x_{n_k}) of (x_n) converges to the same limit.
(iii) Any subsequence (x_{n_k}) of (x_n) is convergent.

Proof. Part $(ii) \Rightarrow (i)$ is clear because the sequence (x_n) is also a subsequence of itself.
For the Part $(i) \Rightarrow (ii)$, assume that $\lim x_n = a \in X$ exists. Let (x_{n_k}) be a subsequence of (x_n). We claim that $\lim x_{n_k} = a$. Let $\varepsilon > 0$. In fact, since $\lim x_n = a$, there is a positive integer N such that $d(a, x_n) < \varepsilon$ for all $n \geq N$. Notice that by the definition of a subsequence, there is a positive integer K such that $n_k \geq N$ for all $k \geq K$. So, we see that $d(a, x_{n_k}) < \varepsilon$ for all $k \geq K$. Thus we have $\lim_{k\to\infty} x_{n_k} = a$.
Part $(ii) \Rightarrow (iii)$ is clear.
It remains to show Part $(iii) \Rightarrow (ii)$. Suppose that there are two subsequences $(x_{n_i})_{i=1}^{\infty}$ and $(x_{m_i})_{i=1}^{\infty}$ converge to distinct limits. Now put $k_1 := n_1$. Choose $m_{i'}$ such that $n_1 < m_{i'}$ and then put $k_2 := m_{i'}$. Then we choose n_i such that $k_2 < n_i$ and put k_3 for such n_i. To repeat the same step, we can get a subsequence $(x_{k_i})_{i=1}^{\infty}$ of (x_n) such that $x_{k_{2i}} = x_{n_{i'}}$ for some $n_{i'}$ and $x_{k_{2i-1}} = x_{m_{j'}}$ for some $m_{j'}$. Since by the assumption $\lim_i x_{n_i} \neq \lim_i x_{m_i}$, $\lim_i x_{k_i}$ does not exist which leads to a contradiction.
The proof is finished. □

We now recall the following important theorem in $\mathbb{R}$ (see [1, Theorem 3.4.8]).

Theorem 2.2. *Every bounded sequence in $\mathbb{R}$ has a convergent subsequence.*

Definition 2.3. *X is said to be compact if for every sequence in X has a convergent subsequence. In particular, a subset A of X is compact if every sequence in A has a convergent subsequence with the limit in A.*

Example 2.4. *(i) Every closed and bounded interval is compact.*
In fact, if (x_n) is any sequence in a closed and bounded interval $[a, b]$, then (x_n) is bounded. Then by Bolzano-Weierstrass Theorem (see [1, Theorem 3.4.8]*), (x_n) has a convergent subsequence (x_{n_k}). Notice that since $a \leq x_{n_k} \leq b$ for all k, then $a \leq \lim_k x_{n_k} \leq b$, and thus $\lim_k x_{n_k} \in [a, b]$. Therefore A is sequentially compact.*
(ii) $(0, 1]$ is not sequentially compact. In fact, if we consider $x_n = 1/n$, then (x_n) is a sequence in $(0, 1]$ but it has no convergent subsequence with the limit sitting in $(0, 1]$.

Proposition 2.5. *Recall that $\ell_\infty^{(N)}$ be the set of all bounded sequences under the* sup*-norm, that is $\|x\|_\infty := \max_{1\leq i\leq N} |x(i)|$ for $x \in \ell_\infty^{(N)}$. If we let S be the unit sphere of $\ell_\infty^{(N)}$, that is $S := \{x \in \ell_\infty^{(N)} : \|x\|_\infty = 1\}$, then S is a compact set.*

Proof. We are going to show for the case of $N = 2$. The proof of the general case is similar. Let (x_n) be a sequence in S. Since $\|x_n\|_\infty = 1$, $|x_n(1)| \leq 1$ for all n, and thus, $(x_n(1))_{n=1}^{\infty}$ is a bounded sequence in $\mathbb{K}$. Then by the Bolzano-Weierstrass Theorem, there is a convergent subsequence $(x_{n_k}(1))$ of $(x_n(1))$. Let $z(1) := \lim_{k\to\infty} x_{n_k}(1) \in \mathbb{K}$. Now we are considering the sequence $(x_{n_k}(2)$. Using the Bolzano-Weierstrass Theorem again, there is a convergent subsequence

$(x_{n_{k_j}}(2))$ of $(x_{n_k}(2))$. Let $z(2) := \lim_{j\to\infty} x_{n_{k_j}}(2)$. Note that $(x_{n_{k_j}}(1))$ is still a subsequence of the sequence $(x_{n_k}(1))$, so we have $z(1) = \lim_{j\to\infty} x_{n_{k_j}}(1)$. If we let $z := (z(1), z(2)) \in \ell_\infty^{(2)}$, then $\lim_{j\to\infty} \|z - x_{n_{k_j}}\|_\infty = 0$. Moreover, we have $\|z\|_\infty = 1$ because

$$||\|z\|_\infty - 1| = |\|z\|_\infty - \|x_{n_{k_j}}\|_\infty| \leq \|z - x_{n_{k_j}}\|_\infty \to 0.$$

Therefore, $(x_{n_{k_j}})$ is a convergent subsequence of (x_n) as desired. The proof is complete. □

Proposition 2.6. *If A is a compact subset of X, then A must be a closed and bounded subset of X.*

Proof. We first claim that A is bounded. Suppose not. We suppose that A is unbounded. If we fix an element $x_1 \in A$, then there is $x_2 \in A$ such that $d(x_1, x_2) > 1$. Using the unboundedness of A, we can find an element x_3 in A such that $d(x_3, x_k) > 1$ for $k = 1, 2$. To repeat the same step, we can find a sequence (x_n) in A such that $d(x_n, x_m) > 1$ for $n \neq m$. Thus A has no convergent subsequence. Thus A must be bounded
Finally, we show that A is closed in X. Let (x_n) be a sequence in A and it is convergent. It needs to show that $\lim_n x_n \in A$. Note that since A is compact, (x_n) has a convergent subsequence (x_{n_k}) such that $\lim_k x_{n_k} \in A$. Then by Proposition 2.1, we see that $\lim_n x_n = \lim_k x_{n_k} \in A$. The proof is finished. □

Corollary 2.7. *Let A be a subset of $\mathbb{R}$. Then A is compact if and only if A is a closed and bounded subset.*

Proof. The necessary part follows from Proposition 2.6 at once.
Now suppose that A is closed and bounded. Let (x_n) be a sequence in A and thus (x_n) is a bounded sequence in $\mathbb{R}$. Then by the Bolzano-Weierstrass Theorem, (x_n) has a subsequence (x_{n_k}) which is convergent in $\mathbb{R}$. Since A is closed, $\lim_k x_{n_k} \in A$. Therefore, A is sequentially compact. □

Remark 2.8. *From Corollary 2.7, we see that the converse of Proposition 2.6 holds when $X = \mathbb{R}$, but it does not hold in general. For example, if $X = \ell^\infty(\mathbb{N})$ and A is the closed unit ball in $\ell^\infty(\mathbb{N})$, that is $A := \{x \in \ell^\infty(\mathbb{N}) : \|x\|_\infty \leq 1\}$, then A is closed and bounded subset of $\ell^\infty(\mathbb{N})$ but it is not sequentially compact. Indeed, if we put $e_n := (e_{n,i})_{i=1}^\infty \in \ell^\infty(\mathbb{N})$, where $e_{n,i} = 1$ as $i = n$; otherwise, $e_{n,i} = 0$. Then (e_n) is a sequence in A but it has no convergent subsequence because $\|e_n - e_m\|_\infty = 2$ for $n \neq m$.*

Definition 2.9. *Let (X, d) and (Y, ρ) be metric spaces. Let $f : X \to Y$ be a function from X into Y. We say that f is continuous at a point $c \in X$ if for any $\varepsilon > 0$, there is $\delta > 0$ such that $\rho(f(x), f(c)) < \varepsilon$ whenever $x \in X$ with $d(x, y) < \delta$.*
Furthermore, f is said to be continuous on A if f is continuous at every point in X.

Remark 2.10. *It is clear that f is continuous at $c \in X$ if and only if for any $\varepsilon > 0$, there is $\delta > 0$ such that $B(c, \delta) \subseteq f^{-1}(B(f(c), \varepsilon))$.*

Proposition 2.11. *With the notation as above, we have*

- *(i) f is continuous at some $c \in X$ if and only if for any sequence $(x_n) \in X$ with $\lim x_n = c$ implies $\lim f(x_n) = f(c)$.*
- *(ii) The following statements are equivalent.*
  - *(ii.a) f is continuous on X.*
  - *(ii.b) $f^{-1}(W) := \{x \in X : f(x) \in W\}$ is open in X for any open subset W of Y.*
  - *(ii.c) $f^{-1}(F) := \{x \in X : f(x) \in F\}$ is closed in X for any closed subset F of Y.*

Proof. Part (i):
Suppose that f is continuous at c. Let (x_n) be a sequence in X with $\lim x_n = c$. We claim that $\lim f(x_n) = f(c)$. In fact, let $\varepsilon > 0$, then there is $\delta > 0$ such that $\rho(f(x), f(c)) < \varepsilon$ whenever $x \in X$ with $d(x, c) < \delta$. Since $\lim x_n = c$, there is a positive integer N such that $d(x_n, c) < \delta$ for $n \geq N$ and hence $\rho(f(x_n), f(c)) < \varepsilon$ for all $n \geq N$. Thus $\lim f(x_n) = f(c)$.
For the converse, suppose that f is not continuous at c. Then we can find $\varepsilon > 0$ such that for any n, there is $x_n \in X$ with $d(x_n, c) < 1/n$ but $\rho(f(x_n), f(c)) \geq \varepsilon$. So, if f is not continuous at c, then there is a sequence (x_n) in X with $\lim x_n = c$ but $(f(x_n))$ does not converge to $f(c)$. Part $(iia) \Leftrightarrow (iib)$:
Suppose that f is continuous on X. Let W be an open subset of Y and $c \in f^{-1}(W)$. Since W is open in Y and $f(c) \in W$, there is $\varepsilon > 0$ such that $B(f(c), \varepsilon) \subseteq W$. Since f is continuous at c, there is $\delta > 0$ such that $B(c, \delta) \subseteq f^{-1}(B(f(c), \varepsilon)) \subseteq f^{-1}(W)$. So $f^{-1}(W)$ is open in X.
It remains to show that the converse of Part (ii). Let $c \in X$. Let $\varepsilon > 0$. Put $W := B(f(c), \varepsilon)$. Then W is an open subset of Y and thus $c \in f^{-1}(W)$ and $f^{-1}(W)$ is open in X. Therefore, there is $\delta > 0$ such that $B(c, \delta) \subseteq f^{-1}(W)$. So, f is continuous at c.
Finally, the last equivalent assertion $(ii.b) \Leftrightarrow (ii.c)$ is clearly from the fact that a subset of a metric space is closed if and only if its complement is open in the given metric space.
The proof is complete. □

Lemma 2.12. *Let A be a compact metric space and let $f : A \to \mathbb{R}$ be a continuous function. If $f(x) > 0$ for all $x \in A$, then there is $c > 0$ such that $f(x) \geq c$ for all $x \in A$.*

Proof. We prove by the contradiction. Assume that for any $c > 0$, there is $x \in A$ such that $f(x) < c$. Considering $c = 1/n$ for $n = 1, 2, ...$ Then for each positive integer n, there is an element $x_n \in A$ such that $0 < f(x_n) < 1/n$ for all $n = 1, 2....$ By the compactness of A, there is convergent subsequence (x_{n_k}) of (x_n) such that $z := \lim_k x_{n_k}$ exists in A. f is continuous on A, so $\lim_k f(x_{n_k}) = f(z)$. Since $0 < f(x_{n_k}) < \frac{1}{n_k}$, we have $f(z) = \lim_k f(x_{n_k}) = 0$. It leads to a contradiction because $f(x) > 0$ for all $x \in A$. The proof is complete. □

3. Lecture 3

Definition 3.1. *We say that two norms $\|\cdot\|$ and $\|\cdot\|'$ on a vector space X are* equivalent*, write $\|\cdot\| \sim \|\cdot\|'$, if there are positive numbers c_1 and c_2 such that $c_1\|\cdot\| \leq \|\cdot\|' \leq c_2\|\cdot\|$ on X.*

Example 3.2. *Consider the norms $\|\cdot\|_1$ and $\|\cdot\|_\infty$ on ℓ^1. We are going to show that $\|\cdot\|_1$ and $\|\cdot\|_\infty$ are not equivalent. In fact, if we put $x_n(i) := (1, 1/2, ..., 1/n, 0, 0, ...)$ for $n, i = 1, 2....$ Then $x_n \in \ell^1$ for all n. Notice that (x_n) is a Cauchy sequence with respect to the norm $\|\cdot\|_\infty$ but it is not a Cauchy sequence with respect to the norm $\|\cdot\|_1$. Hence $\|\cdot\|_1 \nsim \|\cdot\|_\infty$ on ℓ^1.*

Example 3.3. *Recall that the space $L^\infty([0,1])$ is the set of all essential bounded functions defined on $[0,1]$, that is, the set of all $\mathbb{R}$-valued functions f defined on $[0,1]$ such that there is $M > 0$ satisfying the condition: $\lambda\{x \in [0,1] : |f(x)| > M\} = 0$, where λ denotes the Lebesgue measure on $[0,1]$. In this case,*

$$\|f\|_\infty := \inf\{M : \lambda\{x \in [0,1] : |f(x)| > M\} = 0\}.$$

On the other hand, $L^1[0,1]$ denotes the space of all integrable functions on $[0,1]$, that is the set of measurable $\mathbb{R}$-valued functions on $[0,1]$ satisfying the condition:

$$\int_0^1 |f(x)| d\lambda(x) < \infty.$$

Also, we define $\|f\|_1 := \int_0^1 |f(x)|d\lambda(x)$.
It is a known fact that $(L^\infty([0,1]), \|\cdot\|_\infty)$ *and* $(L^1([0,1]), \|\cdot\|_1)$ *both are Banach spaces. (see* [2, Section 9.2]*).*
It is clear that $L^\infty[0,1] \subseteq L^1[0,1]$.
Claim: *The norms* $\|\cdot\|_\infty$ *and* $\|\cdot\|_1$ *are not equivalent on* $L^\infty[0,1]$.
For showing the **Claim**, *it suffices to find a sequence* (f_n) *in* $L^\infty[0,1]$ *that is convergent in* $L^1[0,1]$ *but it is divergent in* $L^\infty[0,1]$.
Now for each positive integer i, *we define a function* $e_i(x)$ *on* $[0,1]$ *by* $e_i(x) \equiv 1$ *if* $x \in (\frac{1}{i+1}, \frac{1}{i})$; *otherwise, set* $e_i(x) \equiv 0$. *Define*

$$f(x) := \sum_{i=1}^{\infty} \sqrt{i} e_i(x)$$

for $x \in [0,1]$. *Notice that* $f \in L^1[0,1]$ *because we have*

$$\int_0^1 |f(x)|d\lambda(x) = \sum_{i=1}^{\infty} \sqrt{i}\lambda(\frac{1}{i+1}, \frac{1}{i}) = \sum_{i=1}^{\infty} \sqrt{i}|\frac{1}{i} - \frac{1}{i+1}| \leq \sum_{i=1}^{\infty} \frac{1}{i^{3/2}} < \infty.$$

On the other hand, for each positive integer n, *let*

$$f_n(x) := \sum_{i=1}^{n} \sqrt{i} e_i(x)$$

for $x \in [0,1]$. *Then each* $f_n \in L^\infty[0,1]$ *and* $\|f_n - f\|_1 \to 0$ *since we have*

$$\|f - f_n\|_1 = \sum_{i=n+1}^{\infty} \sqrt{i}|\frac{1}{i} - \frac{1}{i+1}| \leq \sum_{i=n+1}^{\infty} \frac{1}{i^{3/2}} \to 0 \quad as \quad n \to \infty.$$

However, we note that $f \notin L^\infty[0,1]$, *that is, for each* $M > 0$, *we have* $\lambda\{x \in [0,1] : |f(x)| > M\} > 0$. *Indeed, given any* $M > 0$, *we can find a positive integer* i_0 *such that* $\sqrt{i_0} > M$. *Then by the construction of* f, *we have* $f(x) > M$ *for all* $x \in (\frac{1}{i_0+1}, \frac{1}{i_0})$. *This implies that*

$$\lambda\{x \in [0,1] : |f(x)| > M\} > \frac{1}{i_0(i_0+1)} > 0.$$

Therefore, the sequence (f_n) *must be divergent in* $L^\infty[0,1]$, *otherwise, the limit of* (f_n) *must be* f *that contradicts to* $f \notin L^\infty[0,1]$ *above. So, the sequence* (f_n) *is as required.*

Proposition 3.4. *All norms on a finite dimensional vector space are equivalent.*

Proof. Let X be a finite dimensional vector space and let $\{e_1, ..., e_N\}$ be a vector base of X. For each $x = \sum_{i=1}^N \alpha_i e_i$ for $\alpha_i \in \mathbb{K}$, define $\|x\|_0 = \max\{|a_i| : i = 1, ..., N\}$Then $\|\cdot\|_0$ is a norm X. The result is obtained by showing that all norms $\|\cdot\|$ on X are equivalent to $\|\cdot\|_0$.

Notice that for each $x = \sum_{i=1}^N \alpha_i e_i \in X$, we have $\|x\| \leq (\sum_{i=1}^{N} \|e_i\|)\|x\|_0$. It remains to find $c > 0$ such that $c\|\cdot\|_0 \leq \|\cdot\|$. In fact, let $\mathbb{K}^N$ be equipped with the sup-norm $\|\cdot\|_\infty$, that is $\|(\alpha_1, ..., \alpha_N)\|_\infty = \max_{1 \leq 1 \leq N} |\alpha_i|$. Define a real-valued function f on the unit sphere $S_{\mathbb{K}^N}$ of $\mathbb{K}^N$ by

$$f : (\alpha_1, ..., \alpha_N) \in S_{\mathbb{K}^N} \mapsto \|\alpha_1 e_1 + \cdots + \alpha_n e_N\|.$$

Notice that the map f is continuous and $f > 0$. It is clear that $S_{\mathbb{K}^N}$ is compact with respect to the sup-norm $\|\cdot\|_\infty$ on $\mathbb{K}^N$. Hence, there is $c > 0$ such that $f(\alpha) \geq c > 0$ for all $\alpha \in S_{\mathbb{K}^N}$. This gives $\|x\| \geq c\|x\|_0$ for all $x \in X$ as desired. The proof is finished. □

The following result is clear. The proof is omitted here.

Lemma 3.5. *Let X be a normed space. Then the closed unit ball B_X is compact if and only if every bounded sequence in X has a convergent subsequence.*

Proposition 3.6. *We have the following assertions.*

(i) All finite dimensional normed spaces are Banach spaces. Consequently, any finite dimensional subspace of a normed space must be closed.
(ii) The closed unit ball of any finite dimensional normed space is compact.

Proof. Let $(X, \|\cdot\|)$ be a finite dimensional normed space. With the notation as in the proof of Proposition 3.4 above, we see that $\|\cdot\|$ must be equivalent to the norm $\|\cdot\|_0$. It is clear that X is complete with respect to the norm $\|\cdot\|_0$ and so is complete in the original norm $\|\cdot\|$. The Part (i) follows.
For Part (ii), by using Lemma 3.5, we need to show that any bounded sequence has a convergent subsequence. Let (x_n) be a bounded sequence in X. Since all norms on a finite dimensional normed space are equivalent, it suffices to show that (x_n) has a convergent subsequence with respect to the norm $\|\cdot\|_0$.
Using the notation as in Proposition 3.4, for each x_n, put $x_n = \sum_{k=1}^N \alpha_{n,k} e_k$, $n = 1, 2....$ Then by the definition of the norm $\|\cdot\|_0$, we see that $(\alpha_{n,k})_{n=1}^\infty$ is a bounded sequence in $\mathbb{K}$ for each $k = 1, 2..., N$. Then by the Bolzano-Weierstrass Theorem, for each $k = 1, ..., N$, we can find a convergent subsequence $(\alpha_{n_j,k})_{j=1}^\infty$ of $(\alpha_{n,k})_{n=1}^\infty$. Put $\gamma_k := \lim_{j\to\infty} \alpha_{n_j,k} \in \mathbb{K}$, for $k = 1, .., N$. Put $x := \sum_{k=1}^N \gamma_k e_k$. Then by the definition of the norm $\|\cdot\|_0$, we see that $\|x_{n_j} - x\|_0 \to 0$ as $j \to \infty$. Thus, (x_n) has a convergent subsequence as desired.
The proof is complete. □

In the rest of this section, we are going to show the converse of Proposition 3.6 (ii) also holds. Before showing the main theorem in this section, we need the following useful result.

Lemma 3.7. Riesz's Lemma: *Let Y be a closed proper subspace of a normed space X. Then for each $\theta \in (0, 1)$, there is an element $x_0 \in S_X$ such that $d(x_0, Y) := \inf\{\|x_0 - y\| : y \in Y\} \geq \theta$.*

Proof. Let $u \in X - Y$ and $d := \inf\{\|u - y\| : y \in Y\}$. Notice that since Y is closed, $d > 0$ and hence, we have $0 < d < \frac{d}{\theta}$ because $0 < \theta < 1$. This implies that there is $y_0 \in Y$ such that $0 < d \leq \|u - y_0\| < \frac{d}{\theta}$. Now put $x_0 := \frac{u-y_0}{\|u-y_0\|} \in S_X$. We are going to show that x_0 is as desired. Indeed, let $y \in Y$. Since $y_0 + \|u - y_0\|y \in Y$, we have

$$\|x_0 - y\| = \frac{1}{\|u - y_0\|}\|u - (y_0 + \|u - y_0\|y)\| \geq d/\|u - y_0\| > \theta.$$

So, $d(x_0, Y) \geq \theta$. □

Remark 3.8. *The Riesz's lemma does not hold when $\theta = 1$.*

Theorem 3.9. *Let X be a normed space. Then the following statements are equivalent.*

(i) X is a finite dimensional normed space.
(ii) The closed unit ball B_X of X is compact.
(iii) Every bounded sequence in X has convergent subsequence.

Proof. The implication $(i) \Rightarrow (ii)$ follows from Proposition 3.6 (ii) at once.
Lemma 3.5 gives the implication $(ii) \Rightarrow (iii)$.
Finally, for the implication $(iii) \Rightarrow (i)$, assume that X is of infinite dimension. Fix an element $x_1 \in S_X$. Let $Y_1 = \mathbb{K}x_1$. Then Y_1 is a proper closed subspace of X. The Riesz's lemma gives an

element $x_2 \in S_X$ such that $\|x_1 - x_2\| \geq 1/2$. Now consider $Y_2 = span\{x_1, x_2\}$. Then Y_2 is a proper closed subspace of X since $\dim X = \infty$. To apply the Riesz's Lemma again, there is $x_3 \in S_X$ such that $\|x_3 - x_k\| \geq 1/2$ for $k = 1, 2$. To repeat the same step, there is a sequence $(x_n) \in S_X$ such that $\|x_m - x_n\| \geq 1/2$ for all $n \neq m$. Thus, (x_n) is a bounded sequence but it has no convergent subsequence by using the similar argument as in Proposition 10.2. So, the condition (iii) does not hold if $\dim X = \infty$. The proof is finished. □

4. Lecture 4

Definition 4.1. *Let (X, d) and (Y, ρ) be metric spaces. Let $f : X \to Y$ be a function from X into Y. We say that f is continuous at a point $c \in X$ if for any $\varepsilon > 0$, there is $\delta > 0$ such that $\rho(f(x), f(c)) < \varepsilon$ whenever $x \in X$ with $d(x, y) < \delta$.*
Furthermore, f is said to be continuous on A if f is continuous at every point in X.

Remark 4.2. *It is clear that f is continuous at $c \in X$ if and only if for any $\varepsilon > 0$, there is $\delta > 0$ such that $B(c, \delta) \subseteq f^{-1}(B(f(c), \varepsilon))$.*

Proposition 4.3. *With the notation as above, we have*

- *(i) f is continuous at some $c \in X$ if and only if for any sequence $(x_n) \in X$ with $\lim x_n = c$ implies $\lim f(x_n) = f(c)$.*
- *(ii) The following statements are equivalent.*
  - *(ii.a) f is continuous on X.*
  - *(ii.b) $f^{-1}(W) := \{x \in X : f(x) \in W\}$ is open in X for any open subset W of Y.*
  - *(ii.c) $f^{-1}(F) := \{x \in X : f(x) \in F\}$ is closed in X for any closed subset F of Y.*

Proof. Part (i):
Suppose that f is continuous at c. Let (x_n) be a sequence in X with $\lim x_n = c$. We claim that $\lim f(x_n) = f(c)$. In fact, let $\varepsilon > 0$, then there is $\delta > 0$ such that $\rho(f(x), f(c)) < \varepsilon$ whenever $x \in X$ with $d(x, c) < \delta$. Since $\lim x_n = c$, there is a positive integer N such that $d(x_n, c) < \delta$ for $n \geq N$ and hence $\rho(f(x_n), f(c)) < \varepsilon$ for all $n \geq N$. Thus $\lim f(x_n) = f(c)$.
For the converse, suppose that f is not continuous at c. Then we can find $\varepsilon > 0$ such that for any n, there is $x_n \in X$ with $d(x_n, c) < 1/n$ but $\rho(f(x_n), f(c)) \geq \varepsilon$. So, if f is not continuous at c, then there is a sequence (x_n) in X with $\lim x_n = c$ but $(f(x_n))$ does not converge to $f(c)$. Part $(iia) \Leftrightarrow (iib)$:
Suppose that f is continuous on X. Let W be an open subset of Y and $c \in f^{-1}(W)$. Since W is open in Y and $f(c) \in W$, there is $\varepsilon > 0$ such that $B(f(c), \varepsilon) \subseteq W$. Since f is continuous at c, there is $\delta > 0$ such that $B(c, \delta) \subseteq f^{-1}(B(f(c), \varepsilon)) \subseteq f^{-1}(W)$. So $f^{-1}(W)$ is open in X.
It remains to show that the converse of Part (ii). Let $c \in X$. Let $\varepsilon > 0$. Put $W := B(f(c), \varepsilon)$. Then W is an open subset of Y and thus $c \in f^{-1}(W)$ and $f^{-1}(W)$ is open in X. Therefore, there is $\delta > 0$ such that $B(c, \delta) \subseteq f^{-1}(W)$. So, f is continuous at c.
Finally, the last equivalent assertion $(ii.b) \Leftrightarrow (ii.c)$ is clearly from the fact that a subset of a metric space is closed if and only if its complement is open in the given metric space.
The proof is complete. □

Proposition 4.4. *Let T be a linear operator from a normed space X into a normed space Y. Then the following statements are equivalent.*

- *(i) T is continuous on X.*
- *(ii) T is continuous at $0 \in X$.*
- *(iii) $\sup\{\|Tx\| : x \in B_X\} < \infty$.*

In this case, let $\|T\| = \sup\{\|Tx\| : x \in B_X\}$ and T is said to be bounded.

Proof. $(i) \Rightarrow (ii)$ is obvious.
For $(ii) \Rightarrow (i)$, suppose that T is continuous at 0. Let $x_0 \in X$. Let $\varepsilon > 0$. Then there is $\delta > 0$ such that $\|Tw\| < \varepsilon$ for all $w \in X$ with $\|w\| < \delta$. Therefore, we have $\|Tx - Tx_0\| = \|T(x - x_0)\| < \varepsilon$ for any $x \in X$ with $\|x - x_0\| < \delta$. So, (i) follows.
For $(ii) \Rightarrow (iii)$, since T is continuous at 0, there is $\delta > 0$ such that $\|Tx\| < 1$ for any $x \in X$ with $\|x\| < \delta$. Now for any $x \in B_X$ with $x \neq 0$, we have $\|\frac{\delta}{2}x\| < \delta$. So, we see have $\|T(\frac{\delta}{2}x)\| < 1$ and hence, we have $\|Tx\| < 2/\delta$. So, (iii) follows.
Finally, it remains to show $(iii) \Rightarrow (ii)$. Notice that by the assumption of (iii), there is $M > 0$ such that $\|Tx\| \leq M$ for all $x \in B_X$. So, for each $x \in X$, we have $\|Tx\| \leq M\|x\|$. This implies that T is continuous at 0. The proof is complete. □

Corollary 4.5. *Let $T : X \to Y$ be a bounded linear map. Then we have*

$$\sup\{\|Tx\| : x \in B_X\} = \sup\{\|Tx\| : x \in S_X\} = \inf\{M > 0 : \|Tx\| \leq M\|x\|,\ \forall x \in X\}.$$

Proof. Let $a = \sup\{\|Tx\| : x \in B_X\}$, $b = \sup\{\|Tx\| : x \in S_X\}$ and $c = \inf\{M > 0 : \|Tx\| \leq M\|x\|,\ \forall x \in X\}$.
It is clear that $b \leq a$. Now for each $x \in B_X$ with $x \neq 0$, then we have $b \geq \|T(x/\|x\|)\| = (1/\|x\|)\|Tx\| \geq \|Tx\|$. So, we have $b \geq a$ and thus, $a = b$.
Now if $M > 0$ satisfies $\|Tx\| \leq M\|x\|$, $\forall x \in X$, then we have $\|Tw\| \leq M$ for all $w \in S_X$. So, we have $b \leq M$ for all such M. So, we have $b \leq c$. Finally, it remains to show $c \leq b$. Notice that by the definition of b, we have $\|Tx\| \leq b\|x\|$ for all $x \in X$. So, $c \leq b$. □

Proposition 4.6. *Let X and Y be normed spaces. Let $B(X, Y)$ be the set of all bounded linear maps from X into Y. For each element $T \in B(X, Y)$, let*

$$\|T\| = \sup\{\|Tx\| : x \in B_X\}.$$

be defined as in Proposition 4.4.
Then $(B(X, Y), \|\cdot\|)$ becomes a normed space.
Furthermore, if Y is a Banach space, then so is $B(X, Y)$.
In particular, if $Y = \mathbb{K}$, then $B(X, \mathbb{K})$ is a Banach space. In this case, put $X^* := B(X, \mathbb{K})$ and call it the **dual space** *of X.*

Proof. One can directly check that $B(X, Y)$ is a normed space (**Do It By Yourself!**).
We are going to show that $B(X, Y)$ is complete if Y is a Banach space. Let (T_n) be a Cauchy sequence in $B(X, Y)$. Then for each $x \in X$, it is easy to see that $(T_n x)$ is also a Cauchy sequence in Y. So, $\lim T_n x$ exists in Y for each $x \in X$ because Y is complete. Hence, one can define a map $Tx := \lim T_n x \in Y$ for each $x \in X$. It is clear that T is a linear map from X into Y.
It needs to show that $T \in B(X, Y)$ and $\|T - T_n\| \to 0$ as $n \to \infty$. Let $\varepsilon > 0$. Since (T_n) is a Cauchy sequence in $B(X, Y)$, there is a positive integer N such that $\|T_m - T_n\| < \varepsilon$ for all $m, n \geq N$. So, we have $\|(T_m - T_n)(x)\| < \varepsilon$ for all $x \in B_X$ and $m, n \geq N$. Taking $m \to \infty$, we have $\|Tx - T_n x\| \leq \varepsilon$ for all $n \geq N$ and $x \in B_X$. Therefore, we have $\|T - T_n\| \leq \varepsilon$ for all $n \geq N$. From this, we see that $T - T_N \in B(X, Y)$ and thus, $T = T_N + (T - T_N) \in B(X, Y)$ and $\|T - T_n\| \to 0$ as $n \to \infty$. Therefore, $\lim_n T_n = T$ exists in $B(X, Y)$. □

Proposition 4.7. *Let X and Y be normed spaces. Suppose that X is of finite dimension n. Then we have the following assertions.*

(i) Any linear operator from X into Y must be bounded.
(ii) If $T_k : X \to Y$ is a sequence of linear operators such that $T_k x \to 0$ for all $x \in X$, then $\|T_k\| \to 0$.

Proof. Using Proposition 3.4 and the notation as in the proof, then there is $c > 0$ such that

$$\sum_{i=1}^{n} |\alpha_i| \leq c\| \sum_{i=1}^{n} \alpha_i e_i\|$$

for all scalars $\alpha_1, ..., \alpha_n$. Therefore, for any linear map T from X to Y, we have

$$\|Tx\| \leq \big(\max_{1\leq i\leq n} \|Te_i\|\big) c\|x\|$$

for all $x \in X$. This gives the assertions (i) and (ii) immediately. $\square$

Proposition 4.8. *Let Y be a closed subspace of X and X/Y be the quotient space. For each element $x \in X$, put $\bar{x} := x + Y \in X/Y$ the corresponding element in X/Y. Define*

$$\|\bar{x}\| = \inf\{\|x + y\| : y \in Y\}. \tag{4.1}$$

If we let $\pi : X \to X/Y$ be the natural projection, that is $\pi(x) = \bar{x}$ for all $x \in X$, then $(X/Y, \|\cdot\|)$ is a normed space and π is bounded with $\|\pi\| \leq 1$. In particular, $\|\pi\| = 1$ as Y is a proper closed subspace.
Furthermore, if X is a Banach space, then so is X/Y.
In this case, we call $\|\cdot\|$ in (4.1) the quotient norm on X/Y.

Proof. Notice that since Y is closed, one can directly check that $\|\bar{x}\| = 0$ if and only is $x \in Y$, that is, $\bar{x} = \bar{0} \in X/Y$. It is easy to check the other conditions of the definition of a norm. So, X/Y is a normed space. Also, it is clear that π is bounded with $\|\pi\| \leq 1$ by the definition of the quotient norm on X/Y.
Furthermore, if $Y \subsetneq X$, then by using the Riesz's Lemma 3.7, we see that $\|\pi\| = 1$ at once.
We are going to show the last assertion. Suppose that X is a Banach space. Let $(\bar{x}_n)$ be a Cauchy sequence in X/Y. It suffices to show that $(\bar{x}_n)$ has a convergent subsequence in X/Y by using Lemma 9.2.
Indeed, since $(\bar{x}_n)$ is a Cauchy sequence, we can find a subsequence $(\bar{x}_{n_k})$ of $(\bar{x}_n)$ such that

$$\|\bar{x}_{n_{k+1}} - \bar{x}_{n_k}\| < 1/2^k$$

for all $k = 1, 2....$ Then by the definition of quotient norm, there is an element $y_1 \in Y$ such that $\|x_{n_2} - x_{n_1} + y_1\| < 1/2$. Notice that we have, $\overline{x_{n_1} - y_1} = \bar{x}_{n_1}$ in X/Y. So, there is $y_2 \in Y$ such that $\|x_{n_2} - y_2 - (x_{n_1} - y_1)\| < 1/2$ by the definition of quotient norm again. Also, we have $\overline{x_{n_2} - y_2} = \bar{x}_{n_2}$. Then we also have an element $y_3 \in Y$ such that $\|x_{n_3} - y_3 - (x_{n_2} - y_2)\| < 1/2^2$. To repeat the same step, we can obtain a sequence (y_k) in Y such that

$$\|x_{n_{k+1}} - y_{k+1} - (x_{n_k} - y_k)\| < 1/2^k$$

for all $k = 1, 2....$ Therefore, $(x_{n_k} - y_k)$ is a Cauchy sequence in X and thus, $\lim_k (x_{n_k} - y_k)$ exists in X while X is a Banach space. Set $x = \lim_k (x_{n_k} - y_k)$. On the other hand, notice that we have $\pi(x_{n_k} - y_k) = \pi(x_{n_k})$ for all $k = 1, 2, , ,$. This tells us that $\lim_k \pi(x_{n_k}) = \lim_k \pi(x_{n_k} - y_k) = \pi(x) \in X/Y$ since π is bounded. So, $(\bar{x}_{n_k})$ is a convergent subsequence of $(\bar{x}_n)$ in X/Y. The proof is complete. $\square$

Corollary 4.9. *Let $T : X \to Y$ be a linear map. Suppose that Y is of finite dimension. Then T is bounded if and only if $\ker T := \{x \in X : Tx = 0\}$, the kernel of T, is closed.*

Proof. The necessary part is clear.
Now assume that $\ker T$ is closed. Then by Proposition 4.8, $X/\ker T$ becomes a normed space. Also, it is known that there is a linear injection $\widetilde{T} : X/\ker T \to Y$ such that $T = \widetilde{T} \circ \pi$, where $\pi : X \to X/\ker T$ is the natural projection. Since $\dim Y < \infty$ and $\widetilde{T}$ is injective, $\dim X/\ker T < \infty$. This implies that $\widetilde{T}$ is bounded by Proposition 4.7. Hence T is bounded because $T = \widetilde{T} \circ \pi$ and π is bounded. $\square$

Remark 4.10. The converse of Corollary 4.9 does not hold when Y is of infinite dimension. For example, let $X := \{x \in \ell^2 : \sum_{n=1}^{\infty} n^2|x(n)|^2 < \infty\}$ (notice that X is a vector space **Why?**) and $Y = \ell^2$. Both X and Y are endowed with $\|\cdot\|_2$-norm.
Define $T : X \to Y$ by $Tx(n) = nx(n)$ for $x \in X$ and $n = 1, 2....$ Then T is an unbounded operator(**Check !!**). Notice that $\ker T = \{0\}$ and hence, $\ker T$ is closed. So, the closeness of $\ker T$ does not imply the boundedness of T in general.

We say that two normed spaces X and Y are said to be *isomorphic (resp. isometric isomorphic)* if there is a bi-continuous linear isomorphism (resp. isometric) between X and Y. We also write $X = Y$ if X and Y are isometric isomorphic.

Remark 4.11. *Notice that the inverse of a bounded linear isomorphism may not be bounded.*

Example 4.12. *Let $X : \{f \in C^\infty(-1, 1) : f^{(n)} \in C^b(-1, 1)$ for all $n = 0, 1, 2...\}$ and $Y := \{f \in X : f(0) = 0\}$. Also, X and Y both are equipped with the* sup-*norm $\|\cdot\|_\infty$. Define an operator $S : X \to Y$ by*

$$Sf(x) := \int_0^x f(t)dt$$

for $f \in X$ and $x \in (-1, 1)$. Then S is a bounded linear isomorphism but its inverse S^{-1} is unbounded. In fact, the inverse $S^{-1} : Y \to X$ is given by

$$S^{-1}g := g'$$

for $g \in Y$.

5. Lecture 5

All spaces $X, Y, Z...$ are normed spaces over the field $\mathbb{K}$ throughout this section. By Proposition 4.6, we have the following assertion at once.

Proposition 5.1. *Let X be a normed space. Put $X^* = B(X, \mathbb{K})$. Then X^* is a Banach space and is called the dual space of X.*

Example 5.2. *Let $X = \mathbb{K}^N$. Consider the usual Euclidean norm on X, that is, $\|(x_1, ..., x_N)\| := \sqrt{|x_1|^2 + \cdots |x_N|^2}$. Define $\theta : \mathbb{K}^N \to (\mathbb{K}^N)^*$ by $\theta x(y) = x_1y_1 + \cdots + x_Ny_N$ for $x = (x_1, ..., x_N)$ and $y = (y_1, ..., y_N) \in \mathbb{K}^N$. Notice that $\theta x(y) = \langle x, y\rangle$, the usual inner product on $\mathbb{K}^N$. Then by the Cauchy-Schwarz inequality, it is easy to see that θ is an isometric isomorphism. Therefore, we have $\mathbb{K}^N = (\mathbb{K}^N)^*$.*

Example 5.3. *Define a map $T : \ell^1 \to c_0^*$ by*

$$(Tx)(\eta) = \sum_{i=1}^{\infty} x(i)\eta(i)$$

for $x \in \ell^1$ and $\eta \in c_0$.
Then T is isometric isomorphism and hence, $c_0^* = \ell^1$.

Proof. The proof is divided into the following steps.
Step 1. $Tx \in c_0^*$ for all $x \in \ell^1$.
In fact, let $\eta \in c_0$. Then

$$|Tx(\eta)| \leq |\sum_{i=1}^{\infty} x(i)\eta(i)| \leq \sum_{i=1}^{\infty} |x(i)||\eta(i)| \leq \|x\|_1\|\eta\|_\infty.$$

So, *Step* 1 follows.
Step 2. T is an isometry.
Notice that by *Step* 1, we have $\|Tx\| \leq \|x\|_1$ for all $x \in \ell^1$. It needs to show that $\|Tx\| \geq \|x\|_1$ for all $x \in \ell^1$. Fix $x \in \ell^1$. Now for each $k = 1, 2..$, consider the polar form $x(k) = |x(k)|e^{i\theta_k}$. Notice that $\eta_n := (e^{-i\theta_1}, ..., e^{-i\theta_n}, 0, 0, ...) \in c_0$ for all $n = 1, 2....$ Then we have

$$\sum_{k=1}^{n} |x(k)| = \sum_{k=1}^{n} x(k)\eta_n(k) = Tx(\eta_n) = |Tx(\eta_n)| \leq \|Tx\|$$

for all $n = 1, 2....$ So, we have $\|x\|_1 \leq \|Tx\|$.
Step 3. T is a surjection.
Let $\phi \in c_0^*$ and let $e_k \in c_0$ be given by $e_k(j) = 1$ if $j = k$, otherwise, is equal to 0. Put $x(k) := \phi(e_k)$ for $k = 1, 2...$ and consider the polar form $x(k) = |x(k)|e^{i\theta_k}$ as above. Then we have

$$\sum_{k=1}^{n} |x(k)| = \phi(\sum_{k=1}^{n} e^{-i\theta_k} e_k) \leq \|\phi\| \|\sum_{k=1}^{n} e^{-i\theta_k} e_k\|_\infty = \|\phi\|$$

for all $n = 1, 2....$ Therefore, $x \in \ell^1$.
Finally, we need to show that $Tx = \phi$ and thus, T is surjective. In fact, if $\eta = \sum_{k=1}^{\infty} \eta(k)e_k \in c_0$, then we have

$$\phi(\eta) = \sum_{k=1}^{\infty} \eta(k)\phi(e_k) = \sum_{k=1}^{\infty} \eta(k)x_k = Tx(\eta).$$

So, the proof is finished by the *Steps* 1-3 above. □

Example 5.4. We have the other important examples of the dual spaces.

(i) $(\ell^1)^* = \ell^\infty$.
(ii) For $1 < p < \infty$, $(\ell^p)^* = \ell^q$, where $\frac{1}{p} + \frac{1}{q} = 1$.
(iii) For a locally compact Hausdorff space X, $C_0(X)^* = M(X)$, where $M(X)$ denotes the space of all regular Borel measures on X.

Parts (i) and (ii) can be obtained by the similar argument as in Example 5.3 (see also in [3, Chapter 8]). Part (iii) is known as the *Riesz representation Theorem* which is referred to [3, Section 21.5] for the details.

Example 5.5. Let $C[a, b]$ be the space of all continuous $\mathbb{R}$-valued functions defined on a closed and bounded interval $[a, b]$. Also, the space $C[a, b]$ is endowed with the sup-norm, that is, $\|f\|_\infty := \sup\{|f(x)| : x \in [a, b]$ for $f \in C[a, b]$.
Recall that a function $\rho : [a, b] \to \mathbb{R}$ is said to be a bounded variation if it satisfies the condition:

$$V(\rho) := \sup\{\sum_{k=1}^{n} |\rho(x_k) - \rho(x_{k-1})| : a = x_0 < x_1 < \cdots < x_n = b\} < \infty.$$

Let $BV([a, b])$ denote the space of all bounded variations on $[a, b]$ and let $\|\rho\| := V(\rho)$ for $\rho \in BV([a, b])$. Then $BV([a, b])$ becomes a Banach space.
On the other hand, for $f \in C[a, b]$, the Riemann-Stieltjes integral of f with respect to a bounded variation ρ on $[a, b]$ is defined by

$$\int_a^b f(x)d\rho(x) := \lim_P \sum_{k=1}^{n} f(\xi_k)(\rho(x_k - x_{k-1}),$$

where $P : a = x_0 < x_1 < \cdots < x_n = b$ and $\xi_k \in [x_{k-1}, x_k]$ (**Fact: the Riemann-Stieltjes integral of a continuous function always exists**).

Define a mapping $T : BV([a,b]) \to C[a,b]^*$ by

$$T(\rho)(f) := \int_a^b f(x)d\rho(x)$$

for $\rho \in BV([a,b])$ and $f \in C[a,b]$. Then T is an isometric isomorphism, and hence, we have

$$C[a,b]^* = BV([a,b]).$$

In the rest of this section, we are going to show the Hahn-Banach Theorem which is a very important Theorem in mathematics. Before showing this theorem, we need the following lemma first.

Lemma 5.6. *Let Y be a subspace of X and $v \in X \setminus Y$. Let $Z = Y \oplus \mathbb{K}v$ be the linear span of Y and v in X. If $f \in Y^*$, then there is an extension $F \in Z^*$ of f such that $\|F\| = \|f\|$.*

Proof. We may assume that $\|f\| = 1$ by considering the normalization $f/\|f\|$ if $f \neq 0$.
Case $\mathbb{K} = \mathbb{R}$:
We first note that since $\|f\| = 1$, we have $|f(x) - f(y)| \leq \|(x+v) - (y+v)\|$ for all $x, y \in Y$. This implies that $-f(x) - \|x+v\| \leq -f(y) + \|y+v\|$ for all $x, y \in Y$. Now let $\gamma = \sup\{-f(x) - \|x+v\| : x \in X\}$. This implies that γ exists and

$$-f(y) - \|y+v\| \leq \gamma \leq -f(y) + \|y+v\| \tag{5.1}$$

for all $y \in Y$. We define $F : Z \longrightarrow \mathbb{R}$ by $F(y + \alpha v) := f(y) + \alpha\gamma$. It is clear that $F|_Y = f$. For showing $F \in Z^*$ with $\|F\| = 1$, since $F|_Y = f$ on Y and $\|f\| = 1$, it needs to show $|F(y + \alpha v)| \leq \|y + \alpha v\|$ for all $y \in Y$ and $\alpha \in \mathbb{R}$.
In fact, for $y \in Y$ and $\alpha > 0$, then by inequality 5.1, we have

$$|F(y + \alpha v)| = |f(y) + \alpha\gamma| \leq \|y + \alpha v\|. \tag{5.2}$$

Since y and α are arbitrary in inequality 5.2, we see that $|F(y + \alpha v)| \leq \|y + \alpha v\|$ for all $y \in Y$ and $\alpha \in \mathbb{R}$. Therefore the result holds when $\mathbb{K} = \mathbb{R}$.
Now for the complex case, let $h = \Re e f$ and $g = \Im m f$. Then $f = h + ig$ and f, g both are real linear with $\|h\| \leq 1$. Note that since $f(iy) = if(y)$ for all $y \in Y$, we have $g(y) = -h(iy)$ for all $y \in Y$. This gives $f(\cdot) = h(\cdot) - ih(i\cdot)$ on Y. Then by the real case above, there is a real linear extension H on $Z := Y \oplus \mathbb{R}v \oplus i\mathbb{R}v$ of h such that $\|H\| = \|h\|$. Now define $F : Z \longrightarrow \mathbb{C}$ by $F(\cdot) := H(\cdot) - iH(i\cdot)$. Then $F \in Z^*$ and $F|_Y = f$. Thus it remains to show that $\|F\| = \|f\| = 1$. It needs to show that $|F(z)| \leq \|z\|$ for all $z \in Z$. Note for $z \in Z$, consider the polar form $F(z) = re^{i\theta}$. Then $F(e^{-i\theta}z) = r \in \mathbb{R}$ and thus $F(e^{-i\theta}z) = H(e^{-i\theta}z)$. This yields that

$$|F(z)| = r = |F(e^{-i\theta}z)| = |H(e^{-i\theta}z)| \leq \|H\|\|e^{-i\theta}z\| \leq \|z\|.$$

The proof is finished. □

Remark 5.7. Before completing the proof of the Hahn-Banach Theorem, Let us first recall one of super important results in mathematics, called *Zorn's Lemma*, a very humble name. Every mathematics student should know it.

Zorn's Lemma: Let $\mathcal{X}$ be a non-empty set with a partially order " $\leq$ ". Assume that every totally order subset $\mathcal{C}$ of $\mathcal{X}$ has an upper bound, i.e. there is an element $\mathfrak{z} \in \mathcal{X}$ such that $c \leq \mathfrak{z}$ for all $c \in \mathcal{C}$. Then $\mathcal{X}$ must contain a maximal element $\mathfrak{m}$, that is, if $\mathfrak{m} \leq x$ for some $x \in \mathcal{X}$, then $\mathfrak{m} = x$.

The following is the typical argument of applying the Zorn's Lemma.

Theorem 5.8. Hahn-Banach Theorem : *Let X be a normed space and let Y be a subspace of X. If $f \in Y^*$, then there exists a linear extension $F \in X^*$ of f such that $\|F\| = \|f\|$.*

Proof. Let $\mathcal{X}$ be the collection of the pairs (Y_1, f_1), where $Y \subseteq Y_1$ is a subspace of X and $f_1 \in Y_1^*$ such that $f_1|_Y = f$ and $\|f_1\|_{Y_1^*} = \|f\|_{Y^*}$. Define a partial order $\leq$ on $\mathcal{X}$ by $(Y_1, f_1) \leq (Y_2, f_2)$ if $Y_1 \subseteq Y_2$ and $f_2|_{Y_1} = f_1$. Then by the Zorn's lemma, there is a maximal element $(\widetilde{Y}, F)$ in $\mathcal{X}$. The maximality of $(\widetilde{Y}, F)$ and Lemma 5.6 will give $\widetilde{Y} = X$. The proof is finished. □

Proposition 5.9. *Let X be a normed space and $x_0 \in X$. Then there is $f \in X^*$ with $\|f\| = 1$ such that $f(x_0) = \|x_0\|$. Consequently, we have*

$$\|x_0\| = \sup\{|g(x)| : g \in B_{X^*}\}.$$

Also, if $x, y \in X$ with $x \neq y$, then there exists $f \in X^*$ such that $f(x) \neq f(y)$.

Proof. Let $Y = \mathbb{K}x_0$. Define $f_0 : Y \to \mathbb{K}$ by $f_0(\alpha x_0) := \alpha\|x_0\|$ for $\alpha \in \mathbb{K}$. Then $f_0 \in Y^*$ with $\|f_0\| = \|x_0\|$. Thus, the result follows from the Hahn-Banach Theorem at once. □

Remark 5.10. Proposition 5.9 tells us that the dual space X^* of X must be non-zero. Indeed, the dual space X^* is very "*Large*″ so that it can separate any pair of distinct points in X.
Furthermore, for any normed space Y and any pair of points $x_1, x_2 \in X$ with $x_1 \neq x_2$, we can find an element $T \in B(X, Y)$ such that $Tx_1 \neq Tx_2$. In fact, fix a non-zero element $y \in Y$. Then by Proposition 5.9, there is $f \in X^*$ such that $f(x_1) \neq f(x_2)$. Thus, if we define $Tx = f(x)y$, then $T \in B(X, Y)$ as desired.

Proposition 5.11. *With the notation as above, if M is closed subspace and $v \in X \setminus M$, then there is $f \in X^*$ such that $f(M) \equiv 0$ and $f(v) \neq 0$.*

Proof. Since M is a closed subspace of X, we can consider the quotient space X/M. Let $\pi : X \to X/M$ be the natural projection. Notice that $\bar{v} := \pi(v) \neq 0 \in X/M$ because $\bar{v} \in X \setminus M$. Then by Corollary 5.9, there is a non-zero element $\bar{f} \in (X/M)^*$ such that $\bar{f}(\bar{v}) \neq 0$. Thus, the linear functional $f := \bar{f} \circ \pi \in X^*$ is as desired. □

Recall that a a normed space X is said to be *separable* if there is a countable subset E of X, i.e. E is a finite set or it can be written as a sequence form, $E = \{x_1, x_2, ...\}$, such that for each element $a \in X$ and a positive number $r > 0$, we can find an element $x_n \in E$ such that $\|a - x_n\| < r$.
For example, the set of all rational numbers is a dense subset of $\mathbb{R}$, thus, $\mathbb{R}$ is separable. The followings are important examples of separable Banach spaces:

Example 5.12. *c_0; ℓ_p for $1 \leq p < \infty$ and $C[a, b]$ are separable Banach spaces.*
However, ℓ_∞ is non-separable.

Definition 5.13. *A sequence of element $(e_n)_{n=1}^\infty$ in a normed space X is called a Schauder base for X if for each element $x \in X$, there is a unique sequence of scalars (α_n) such that*

$$x = \sum_{n=1}^{\infty} \alpha_n e_n. \tag{5.3}$$

Note: The expression in Eq. 5.3 depends on the order of e_n's.

Remark 5.14. Note that if X has a Scahuder base, then X must be separable. The following natural question was first raised by Banach (1932).
The base problem: Does every separable Banach space have a Schauder base?
The answer is "**No**″!
This problem was completely solved by P. Enflo in 1973.

Proposition 5.15. *Using the notation as above, if X^* is separable, then X is separable.*

Proof. Let $F := \{f_1, f_2....\}$ be a dense subset of X^*. Then there is a sequence (x_n) in X with $\|x_n\| = 1$ and $|f_n(x_n)| \geq 1/2\|f_n\|$ for all n. Now let M be the closed linear span of x_n's. Then M is a separable closed subspace of X. We are going to show that $M = X$.
Suppose not. Proposition 5.11 will give us a non-zero element $f \in X^*$ such that $f(M) \equiv 0$. From this, we first see that $f \neq f_m$ for all $m = 1, 2...$ because $f(x_m) = 0$ and $f_m(x_m) \neq 0$ for all $m = 1, 2...$ Also, notice that $B(f, r) \cap F$ must be infinite for all $r > 0$. Thus, there is a subsequence (f_{n_k}) such that $\|f_{n_k} - f\| \to 0$. This gives

$$\frac{1}{2}\|f_{n_k}\| \leq |f_{n_k}(x_{n_k})| = |f_{n_k}(x_{n_k}) - f(x_{n_k})| \leq \|f_{n_k} - f\| \to 0$$

because $f(M) \equiv 0$. So $\|f_{n_k}\| \to 0$ and hence $f = 0$. It leads to a contradiction again. Thus, we can conclude that $M = X$ as desired. □

Remark 5.16. The converse of Proposition 5.15 does not hold. For example, consider $X = \ell^1$. Then ℓ^1 is separable but the dual space $(\ell^1)^* = \ell^\infty$ is not.

Proposition 5.17. *Let X and Y be normed spaces. For each element $T \in B(X, Y)$, define a linear operator $T^* : Y^* \to X^*$ by*

$$T^*y^*(x) := y^*(Tx)$$

for $y^* \in Y^*$ and $x \in X$. Then $T^* \in B(Y^*, X^*)$ and $\|T^*\| = \|T\|$. In this case, T^* is called the adjoint operator of T.

Proof. We first claim that $\|T^*\| \leq \|T\|$ and hence, $\|T^*\|$ is bounded.
In fact, for any $y^* \in Y^*$ and $x \in X$, we have $|T^*y^*(x)| = |y^*(Tx)| \leq \|y^*\|\|T\|\|x\|$. Thus, $\|T^*y^*\| \leq \|T\|\|y^*\|$ for all $y^* \in Y^*$. Thus, $\|T^*\| \leq \|T\|$.
It remains to show $\|T\| \leq \|T^*\|$. Let $x \in B_X$. Then by Proposition 5.9, there is $y^* \in S_{X^*}$ such that $\|Tx\| = |y^*(Tx)| = |T^*y^*(x)| \leq \|T^*y^*\| \leq \|T^*\|$. This implies that $\|T\| \leq \|T^*\|$. □

Example 5.18. Let X and Y be the finite dimensional normed spaces. Let $(e_i)_{i=1}^n$ and $(f_j)_{j=1}^m$ be the bases for X and Y respectively. Let $\theta_X : X \to X^*$ and $\theta_Y : X \to Y^*$ be the identifications as in Example 5.2. Let $e_i^* := \theta_X e_i \in X^*$ and $f_j^* := \theta_Y f_j \in Y^*$. Then $e_i^*(e_l) = \delta_{il}$ and $f_j^*(f_l) = \delta_{jl}$, where, $\delta_{il} = 1$ if $i = l$; otherwise is 0.
Now if $T \in B(X, Y)$ and $(a_{ij})_{m\times n}$ is the representative matrix of T corresponding to the bases $(e_i)_{i=1}^n$ and $(f_j)_{j=1}^m$ respectively, then $a_{kl} = f_k^*(Te_l) = T^*f_k^*(e_l)$. Therefore, if $(a'_{lk})_{n\times m}$ is the representative matrix of T^* corresponding to the bases (f_j^*) and (e_i^*), then $a_{kl} = a'_{lk}$. Hence the transpose $(a_{kl})^t$ is the the representative matrix of T^*.

Definition 5.19. *Let X be a normed space. A sequence (x_n) is said to be weakly convergent if there is $x \in X$ such that $f(x_n) \to f(x)$ for all $f \in X^*$. In this case, x is called a weak limit of (x_n).*

Proposition 5.20. *A weak limit of a sequence is unique if it exists. In this case, if (x_n) weakly converges to x, write $x = w\text{-}\lim_n x_n$ or $x_n \xrightarrow{w} x$.*

Proof. The uniqueness follows from the Hahn-Banach Theorem immediately. □

Remark 5.21. *It is clear that if a sequence (x_n) converges to $x \in X$ in norm, then $x_n \xrightarrow{w} x$. However, the weakly convergence of a sequence does not imply the norm convergence.*
For example, consider $X = c_0$ and (e_n). Then $f(e_n) \to 0$ for all $f \in c_0^* = \ell^1$ but (e_n) is not convergent in c_0.

6. Lecture 6

Throughout this section, let X and Y be normed spaces.
Recall that a subset V of X is said to be open if for each element $x \in V$, there is $r > 0$ such that $B(x, r) \subseteq V$.

Definition 6.1. *A linear map $T : X \to Y$ is called an open map if $T(V)$ is an open subset of Y whenever V is an open subset of Y.*

The following theorem is one of important theorems in Functional Analysis.

Theorem 6.2. Open Mapping Theorem *Suppose that X and Y both are Banach spaces. If T is a bounded linear surjection from X onto Y, then T is an open map.*

Remark 6.3. *Example 4.12 shows that the assumption of the completeness of X and Y in the Open Mapping Theorem is essential.*

Corollary 6.4. *Let X and Y be Banach spaces. If $T : X \to Y$ is a bounded linear isomorphism, then the inverse $T^{-1} : Y \to X$ is also bounded.*

Proof. The assertion follows from the Open Mapping Theorem. □

Corollary 6.5. *Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be the complete norms on a vector space E. Suppose that there is $c > 0$ such that $\|\cdot\|_2 \leq c\|\cdot\|_1$ on E. Then $\|\cdot\|_1 \sim \|\cdot\|_2$.*

Proof. Notice that the identity map $I : (E, \|\cdot\|_1) \to (E, \|\cdot\|_2)$ is a bounded isomorphism by the assumption. Then the result follows from Corollary 6.4 immediately. □

7. Lecture 7

In this section, we are going to investigate the applications of Open Mapping Theorem. The following result is one of the important theorems in functional analysis.
Let $T : X \to Y$ be a linear map from a normed space X into a normed space Y. The graph of T, write $G(T)$, is defined by the following

$$G(T) := \{(x, Tx) : x \in X\} (\subseteq X \times Y).$$

Definition 7.1. *With the notation as above, an operator $T : X \to Y$ is said to be closed if the graph $G(T)$ of T is closed in the following sense:*
if (x_n) is a convergent sequence in X with $\lim_n x_n = x \in X$ such that $\lim Tx_n = y \in Y$ exists, then $Tx = y$.

The following result is clear.

Proposition 7.2. *Every bounded linear operator must be closed.*

Remark 7.3. *The following example shows that the converse of 7.2 does not hold.*

Example 7.4. *Let $X := \{f : (-1, 1) \to \mathbb{R} : f^{(n)}$ exists and bounded for all $n = 0, 1, ..\}$. X is equipped with the sup-norm $\|\cdot\|_\infty$. Define $T : X \to X$ by $Tf = f'$. Then T is closed but it is not bounded.*

Theorem 7.5. Closed Graph Theorem *Let X and Y be Banach spaces. Let $T : X \to Y$ be a linear operator. Then T is bounded if and only if T is closed.*

Proof. The necessary condition is clear. For showing the sufficient condition, now we assume that T is closed. We first define a norm $\|\cdot\|_0$ on X by

$$\|x\|_0 := \|x\| + \|Tx\|$$

for $x \in X$.
Claim 1: X is complete in the norm $\|\cdot\|_0$.
In fact, it is clear that if (x_n) is a Cauchy sequence in X with respect to the new norm $\|\cdot\|_0$, then so are the sequences (x_n) and (Tx_n) with respect to the original norm in X and Y respectively. Since X and Y both are Banach spaces, we see that $\lim_n x_n = x$ (in the original norm of $\|\cdot\|$ on X) and $\lim Tx_n = y$ both exist in X and Y respectively. From this we see that $Tx = y$ by the closeness of T. Thus, we have

$$\|x_n - x\|_0 = \|x_n - x\| + \|Tx_n - Tx\| = \|x_n - x\| + \|Tx_n - y\| \to 0 \text{ as } n \to \infty.$$

Therefore $\|\cdot\|_0$ is a complete norm on X. The Claim 1 follows.
On the other hand, we have $\|\cdot\| \leq \|\cdot\|_0$ on X. Then by Corollary 6.5 and Claim 1, we see that $\|\cdot\| \sim \|\cdot\|_0$ on X and thus, there is $c > 0$ such that $\|\cdot\|_0 \leq c\|\cdot\|$ on X. Therefore, we have $\|Tx\| \leq \|x\|_0 \leq c\|x\|$ for all $x \in X$. Hence, T is bounded. □

Proposition 7.6. *Let E and F be the closed subspaces of a Banach space X such that $X = E \oplus F$. Define an operator $P : X \to X$ by $Px = u$ if $x = u + v$ for $u \in E$ and $v \in F$ (in this case, P is called the projection along the decomposition $X = E \oplus F$). Then P is bounded.*

Proof. Suppose that (x_n) is a convergent sequence in X with the limit $x \in X$ such that $\lim Px_n = y \in X$. Put $x_n = u_n + v_n$ and $x = u + v$ for $u_n, u \in E$ and $v_n, v \in F$. Since $u_n = Px_n \to y$ and E is closed, we have $y \in E$. This implies that $v_n = x_n - u_n \to x - y$. From this we have $x - y \in F$ because $v_n \in F$ and F is closed. This implies that $Px = y$. The Closed Graph Theorem will implies that P is bounded as desired. □

Theorem 7.7. Uniform Boundedness Theorem : *Let $\{T_i : X \longrightarrow Y : i \in I\}$ be a family of bounded linear operators from a Banach space X into a normed space Y. Suppose that for each $x \in X$, we have $\sup\limits_{i\in I} \|T_i(x)\| < \infty$. Then $\sup\limits_{i\in I} \|T_i\| < \infty$.*

Proof. For each $x \in X$, define

$$\|x\|_0 := \max(\|x\|, \sup_{i\in I} \|T_i(x)\|).$$

Then $\|\cdot\|_0$ is a norm on X and $\|\cdot\| \leq \|\cdot\|_0$ on X. If $(X, \|\cdot\|_0)$ is complete, then by the Open Mapping Theorem. This implies that $\|\cdot\|$ is equivalent to $\|\cdot\|_0$ and thus there is $c > 0$ such that

$$\|T_j(x)\| \leq \sup_{i\in I} \|T_i(x)\| \leq \|x\|_0 \leq c\|x\|$$

for all $x \in X$ and for all $j \in I$. So $\|T_j\| \leq c$ for all $j \in I$ is as desired.
Thus it remains to show that $(X, \|\cdot\|_0)$ is complete. In fact, if (x_n) is a Cauchy sequence in $(X, \|\cdot\|_0)$, then it is also a Cauchy sequence with respect to the norm $\|\cdot\|$ on X. Write $x := \lim_n x_n$ with respect to the norm $\|\cdot\|$. Also for any $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that $\|T_i(x_n - x_m)\| < \varepsilon$ for all $m, n \geq N$ and for all $i \in I$. Now fixing $i \in I$ and $n \geq N$ and taking $m \to \infty$, we have $\|T_i(x_n - x)\| \leq \varepsilon$ and thus $\sup_{i\in I} \|T_i(x_n - x)\| \leq \varepsilon$ for all $n \geq N$. So we have $\|x_n - x\|_0 \to 0$ and hence $(X, \|\cdot\|_0)$ is complete. The proof is finished. □

Remark 7.8. *Consider* $c_{00} := \{\mathbf{x} = (x_n) : \exists\ N, \forall\ n \geq N; x_n \equiv 0\}$ *which is endowed with* $\|\cdot\|_\infty$. *Now for each* $k \in \mathbb{N}$, *if we define* $T_k \in c_{00}^*$ *by* $T_k((x_n)) := kx_k$, *then* $\sup_k |T_k(\mathbf{x})| < \infty$ *for each* $\mathbf{x} \in c_{00}$ *but* $(\|T_k\|)$ *is not bounded, in fact,* $\|T_k\| = k$. *Thus the assumption of the completeness of* X *in Theorem 7.7 is essential.*

Corollary 7.9. *Let* X *and* Y *be as in Theorem 7.7. Let* $T_k : X \longrightarrow Y$ *be a sequence of bounded operators. Assume that* $\lim_k T_k(x)$ *exists in* Y *for all* $x \in X$. *Then there is* $T \in B(X,Y)$ *such that* $\lim_k \|(T - T_k)x\| = 0$ *for all* $x \in X$. *Moreover, we have* $\|T\| \leq \liminf_k \|T_k\|$.

Proof. Notice that by the assumption, we can define a linear operator T from X to Y given by $Tx := \lim_k T_k x$ for $x \in X$. It needs to show that T is bounded. In fact, $(\|T_k\|)$ is bounded by the Uniform Boundedness Theorem since $\lim_k T_k x$ exists for all $x \in X$. So for each $x \in B_X$, there is a positive integer K such that $\|Tx\| \leq \|T_K x\| + 1 \leq (\sup_k \|T_k\|) + 1$. Thus, T is bounded.
Finally, it remains to show the last assertion. In fact, notice that for any $x \in B_X$ and $\varepsilon > 0$, there is $N(x) \in \mathbb{N}$ such that $\|Tx\| < \|T_k x\| + \varepsilon < \|T_k\| + \varepsilon$ for all $k \geq N(x)$. This gives $\|Tx\| \leq \inf_{k \geq N(x)} \|T_k\| + \varepsilon$ for all $k \geq N(x)$ and hence, $\|Tx\| \leq \inf_{k \geq N(x)} \|T_k\| + \varepsilon \leq \sup_n \inf_{k \geq n} \|T_k\| + \varepsilon$ for all $x \in B_X$ and $\varepsilon > 0$. Thus, we have $\|T\| \leq \liminf_k \|T_k\|$ as desired. □

8. Lecture 8

From now on, all vectors spaces are over the complex field. Recall that an *inner product* on a vector space V is a function $(\cdot,\cdot) : V \times V \to \mathbb{C}$ which satisfies the following conditions.

(i) $(x,x) \geq 0$ for all $x \in V$ and $(x,x) = 0$ if and only if $x = 0$.
(ii) $\overline{(x,y)} = (y,x)$ for all $x, y \in V$.
(iii) $(\alpha x + \beta y, z) = \alpha(x,z) + \beta(y,z)$ for all $x, y, z \in V$ and $\alpha, \beta \in \mathbb{C}$.

Consequently, for each $x \in V$, the map $y \in V \mapsto (x,y) \in \mathbb{C}$ is conjugate linear by the conditions (ii) and (iii), that is $(x, \alpha y + \beta z) = \bar{\alpha}(x,y) + \bar{\beta}(x,z)$ for all $y, z \in V$ and $\alpha, \beta \in \mathbb{C}$.
Also, the inner product $(\cdot,\cdot)$ will give a norm on V which is defined by

$$\|x\| := \sqrt{(x,x)}$$

for $x \in V$.

We first recall the following useful properties of an inner product space which can be found in the standard text books of linear algebras.

Proposition 8.1. *Let V be an inner product space. For all $x, y \in V$, we always have:*

(i): (**Cauchy-Schwarz inequality**)*: $|(x,y)| \leq \|x\|\|y\|$ Consequently, the inner product on $V \times V$ is jointly continuous.*

(ii): (**Parallelogram law**)*: $\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$*

Furthermore, a norm $\|\cdot\|$ on a vector space X is induced by an inner product if and only if it satisfies the Parallelogram law. In this case such inner product is given by the following:

$$\mathfrak{Re}(x,y) = \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2) \quad \textit{and} \quad \mathfrak{Im}(x,y) = \frac{1}{4}(\|x+iy\|^2 - \|x-iy\|^2)$$

for all $x, y \in X$.

Proof. $[(i)]$: Let $x, y \in V$ with $y \neq 0$ and $\lambda \in \mathbb{C}$. We first notice that if $\lambda = \frac{(x,y)}{(y,y)}$, then $(x - \lambda y, y) = 0$. In this case, we see that

$$\|x\|^2 = \|x - \lambda y\|^2 + \|\lambda y\|^2 \geq |\lambda|^2\|y\|^2 = \frac{|(x,y)|^2}{\|y\|^2}.$$

Part (i) follows. □

Proposition 8.2. *Let V be an inner product space. Then the inner product $(\cdot,\cdot) : V \times V \to \mathbb{C}$ is jointly continuous, that is, $(x_n, y_n) \to (x,y)$ whenever (x_n) and (y_n) both are convergent sequences in V with the limits x and y respectively.*

Proof. We first note that (x_n) is bounded because (x_n) is convergent. Then by using Cauchy-Schwarz inequality, we have

$$|(x_n, y_n) - (x,y)| \leq |(x_n, y_n) - (x_n, y)| + |(x_n, y) - (x,y)| \leq \|x_n\|\|y_n - y\| + \|y\|\|x_n - x\|$$

for all n. This gives the result immediately. □

Example 8.3. *It follows from Proposition 12.1 immediately that ℓ^2 is an inner product space and ℓ^p is not for all $p \in [1,\infty] \setminus \{2\}$.*

From now on, all vector spaces are assumed to be a complex inner product spaces. Recall that two vectors x and y in an inner product space V are said to be *orthogonal* if $(x,y) = 0$.
Also, a set of elements $\{v_i\}_{i \in I}$ in V is said to be *orthonormal* if $(x_i, x_i) = 1$ and $(x_i, x_j) = 0$ for $i, j \in I$ with $i \neq j$. The following is known in a standard course of linear algebra.

Proposition 8.4. Gram-Schmidt process *Let $\{x_1, x_2, ...\}$ be a sequence of linearly independent vectors in an inner product space V. Put $e_1 := x_1/\|x_1\|$. Define e_n inductively on n by*

$$e_{n+1} := \frac{x_n - \sum_{k=1}^{n}(x, e_k)e_k}{\|x_n - \sum_{k=1}^{n}(x, e_k)e_k\|}.$$

Then $\{e_n : n = 1, 2, ..\}$ forms an orthonormal system in V Moreover, the linear span of $x_1, ..., x_n$ is equal to the linear span of $e_1, ..., e_n$ for all $n = 1, 2....$

Proposition 8.5. (**Bessel′s inequality**) : *Let $\{e_1, ..., e_N\}$ be an orthonormal set in an inner product space V, that is $(e_i, e_j) = 1$ if $i = j$, otherwise is equal to 0. Then for any $x \in V$, we have*

$$\sum_{i=1}^{N} |(x, e_i)|^2 \leq \|x\|^2.$$

Proof. It can be obtained by the following equality immediately

$$\|x - \sum_{i=1}^{N}(x, e_i)e_i\|^2 = \|x\|^2 - \sum_{i=1}^{N} |(x, e_i)|^2.$$

□

Corollary 8.6. *Let $(e_i)_{i\in I}$ be an orthonormal set in an inner product space V. Then for any element $x \in V$, the set*

$$\{i \in I : (e_i, x) \neq 0\}$$

is countable.

Proof. Note that for each $x \in V$, we have

$$\{i \in I : (e_i, x) \neq 0\} = \bigcup_{n=1}^{\infty}\{i \in I : |(e_i, x)| \geq 1/n\}.$$

Then the Bessel's inequality implies that the set $\{i \in I : |(e_i, x)| \geq 1/n\}$ must be finite for each $n \geq 1$. So the result follows. □

The following is one of the most important classes in mathematics.

Definition 8.7. *A Hilbert space is a Banach space whose norm is given by an inner product.*

In the rest of this section, X always denotes a complex Hilbert space with an inner product $(\cdot, \cdot)$.

Proposition 8.8. *Let (e_n) be a sequence of orthonormal vectors in a Hilbert space X. Then for any $x \in V$, the series $\sum_{n=1}^{\infty}(x, e_n)e_n$ is convergent.*
Moreover, if $(e_{\sigma(n)})$ is a rearrangement of (e_n), that is, $\sigma : \{1, 2...\} \longrightarrow \{1, 2, ..\}$ is a bijection. Then we have

$$\sum_{n=1}^{\infty}(x, e_n)e_n = \sum_{n=1}^{\infty}(x, e_{\sigma(n)})e_{\sigma(n)}.$$

Proof. Since X is a Hilbert space, the convergence of the series $\sum_{n=1}^{\infty}(x, e_n)e_n$ follows from the Bessel's inequality at once. In fact, if we put $s_p := \sum_{n=1}^{p}(x, e_n)e_n$, then we have

$$\|s_{p+k} - s_p\|^2 = \sum_{p+1\leq n\leq p+k} |(x, e_n)|^2.$$

Now put $y = \sum_{n=1}^{\infty}(x, e_n)e_n$ and $z = \sum_{n=1}^{\infty}(x, e_{\sigma(n)})e_{\sigma(n)}$. Notice that we have

$$\begin{aligned}(y, y - z) &= \lim_N(\sum_{n=1}^{N}(x, e_n)e_n, \sum_{n=1}^{N}(x, e_n)e_n - z)\\ &= \lim_N \sum_{n=1}^{N}|(x, e_n)|^2 - \lim_N \sum_{n=1}^{N}(x, e_n)\sum_{j=1}^{\infty}\overline{(x, e_{\sigma(j)})}(e_n, e_{\sigma(j)})\\ &= \sum_{n=1}^{\infty}|(x, e_n)|^2 - \lim_N \sum_{n=1}^{N}(x, e_n)\overline{(x, e_n)} \quad \text{(N.B: for each } n\text{, there is a unique } j \text{ such that } n = \sigma(j))\\ &= 0.\end{aligned}$$

Similarly, we have $(z, y - z) = 0$. The result follows. □

A family of an orthonormal vectors, say $\mathcal{B}$, in X is said to be **complete** if it is maximal with respect to the set inclusion order,that is, if $\mathcal{C}$ is another family of orthonormal vectors with $\mathcal{B} \subseteq \mathcal{C}$, then $\mathcal{B} = \mathcal{C}$.
A complete orthonormal subset of X is also called an **orthonormal base** of X.

Proposition 8.9. *Let $\{e_i\}_{i\in I}$ be a family of orthonormal vectors in X. Then the followings are equivalent:*
- *(i): $\{e_i\}_{i\in I}$ is complete;*
- *(ii): if $(x, e_i) = 0$ for all $i \in I$, then $x = 0$;*
- *(iii): for any $x \in X$, we have $x = \sum_{i\in I}(x, e_i)e_i$;*
- *(iv): for any $x \in X$, we have $\|x\|^2 = \sum_{i\in I}|(x, e_i)|^2$.*

In this case, the expression of each element $x \in X$ in Part (iii) is unique.

Note : *there are only countable many $(x, e_i) \neq 0$ by Corollary 12.5, so the sums in (iii) and (iv) are convergent by Proposition 12.7.*

Proposition 8.10. *Let X be a Hilbert space. Then*
- *(i) : X processes an orthonormal base.*
- *(ii) : If $\{e_i\}_{i\in I}$ and $\{f_j\}_{j\in J}$ both are the orthonormal bases for X, then I and J have the same cardinality, that is, there is a bijection from I onto J. In this case, the cardinality $|I|$ of I is called the orthonormal dimension of X.*

Proof. Part (i) follows from Zorn's Lemma at once.
For part (ii), if the cardinality $|I|$ is finite, then the assertion is clear since $|I| = \dim X$ (*vector space dimension*) in this case.
Now assume that $|I|$ is infinite, for each e_i, put $J_{e_i} := \{j \in J : (e_i, f_j) \neq 0\}$. Note that since $\{e_i\}_{i\in I}$ is maximal, Proposition 12.8 implies that we have

$$\{f_j\}_{j\in J} \subseteq \bigcup_{i\in I} J_{e_i}.$$

Notice that J_{e_i} is countable for each e_i by using Proposition 12.5. On the other hand, we have $|\mathbb{N}| \leq |I|$ because $|I|$ is infinite and thus $|\mathbb{N} \times I| = |I|$. Then we have

$$|J| \leq \sum_{i\in I}|J_{e_i}| = \sum_{i\in I}|\mathbb{N}| = |\mathbb{N} \times I| = |I|.$$

From symmetry argument, we also have $|I| \leq |J|$. □

Remark 8.11. *Recall that a vector space dimension of X is defined by the cardinality of a maximal linearly independent set in X.*
Notice that if X is finite dimensional, then the orthonormal dimension is the same as the vector space dimension.
Also, the vector space dimension is larger than the orthornormal dimension in general since every orthogonal set must be linearly independent.

Example 8.12. The followings are important classes of Hilbert spaces.

(i) $\mathbb{C}^n$ is a n-dimensional Hilbert space. In this case , the inner product is given by $(z, w) := \sum_{k=1}^{n} z_k \overline{w}_k$ for $z = (z_1, ..., z_n)$ and $(w_1, ..., w_n)$ in $\mathbb{C}^n$.
The natural basis $\{e_1, ..., e_n\}$ forms an orthonormal basis for $\mathbb{C}^n$.

(ii) ℓ^2 is a separable Hilbert space of infinite dimension whose inner product is given by $(x, y) := \sum_{n=1}^{\infty} x(n)\overline{y(n)}$ for $x, y \in \ell^2$.
If we put $e_n(n) = 1$ and $e_n(k) = 0$ for $k \neq n$, then $\{e_n\}$ is an orthonormal basis for ℓ^2.

(iii) Let $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$. For each $f \in C(\mathbb{T})$ (the space of all complex-valued continuous functions defined on $\mathbb{T}$), the integral of f is defined by

$$\int_{\mathbb{T}} f(z)dz := \frac{1}{2\pi}\int_0^{2\pi} f(e^{it})dt = \frac{1}{2\pi}\int_0^{2\pi} \mathfrak{Re} f(e^{it})dt + \frac{i}{2\pi}\int_0^{2\pi} \mathfrak{Im} f(e^{it})dt.$$

An inner product on $C(\mathbb{T})$ is given by

$$(f, g) := \int_{\mathbb{T}} f(z)\overline{g(z)}dz$$

for each $f, g \in C(\mathbb{T})$. We write $\|\cdot\|_2$ for the norm induced by this inner product.
The Hilbert space $L^2(\mathbb{T})$ is defined by the completion of $C(\mathbb{T})$ under the norm $\|\cdot\|_2$.
Now for each $n \in \mathbb{Z}$, put $f_n(z) = z^n$. We claim that $\{f_n : n = 0, \pm1, \pm2, ...\}$ is an orthonormal basis for $L^2(\mathbb{T})$.
In fact, by using the Euler Formula: $e^{i\theta} = \cos\theta + i \sin\theta$ for $\theta \in \mathbb{R}$, we see that the family $\{f_n : n \in \mathbb{Z}\}$ is orthonormal.
It remains to show that the family $\{f_n\}$ is maximal. By Proposition 12.9, it needs to show that if $(g, f_n) = 0$ for all $n \in \mathbb{Z}$, then $g = 0$ in $L^2(\mathbb{T})$. for showing this, we have to make use the known fact that every element in $L^2(\mathbb{T})$ can be approximated by the polynomial functions on $\mathbb{Z}$ in $\|\cdot\|_2$-norm due to the *the Stone-Weierstrass Theorem*. Thus, we can find a sequence of polynomials (p_n) such that $\|g - p_n\|_2 \to 0$ as $n \to 0$. Since $(g, f_n) = 0$ for all n, we see that $(g, p_n) = 0$ for all n. Therefore, we have

$$\|g\|_2^2 = \lim_n (g, p_n) = 0.$$

The proof is finished.

We say that two Hilbert spaces X and Y are said to be *isomorphic* if there is linear isomorphism U from X onto Y such that $(Ux, Ux') = (x, x')$ for all $x, x' \in X$. In this case U is called a *unitary operator.*

Theorem 8.13. *Two Hilbert spaces are isomorphic if and only if they have the same orthonornmal dimension.*

Proof. The converse part ($\Leftarrow$) is clear.
Now for the ($\Rightarrow$) part, let X and Y be isomorphic Hilbert spaces. Let $U : X \longrightarrow Y$ be a unitary. Note that if $\{e_i\}_{i \in I}$ is an orthonormal base of X, then $\{Ue_i\}_{i \in I}$ is also an orthonormal base of Y. Thus the necessary part follows from Proposition 12.9 at once. $\square$

Corollary 8.14. *The Hilbert spaces $L^2(\mathbb{T})$ and ℓ^2 are isomorphic.*

Proof. In Examples 12.11 (ii) and (iii), we have shown that the Hilbert spaces $L^2(\mathbb{T})$ and ℓ^2 have the same orthonormal dimension. Then by Theorem 12.14 above, we see that $L^2(\mathbb{T})$ and ℓ^2 are isomorphic. □

Corollary 8.15. *Every separable Hilbert space is isomorphic to ℓ^2 or $\mathbb{C}^n$ for some n.*

Proof. Let X be a separable Hilbert space.
If $\dim X < \infty$, then it is clear that X is isomorphic to $\mathbb{C}^n$ for $n = \dim X$.
Now suppose that $\dim X = \infty$ and its orthonormal dimension is larger than $|\mathbb{N}|$, that is X has an orthonormal base $\{f_i\}_{i\in I}$ with $|I| > |\mathbb{N}|$. Note that since $\|f_i - f_j\| = \sqrt{2}$ for all $i, j \in I$ with $i \neq j$. This implies that $B(e_i, 1/4) \cap B(e_j, 1/4) = \emptyset$ for $i \neq j$.
On the other hand, if we let D be a countable dense subset of X, then $B(f_i, 1/4) \cap D \neq \emptyset$ for all $i \in I$. So for each $i \in I$, we can pick up an element $x_i \in D \cap B(f_i, 1/4)$. Therefore, one can define an injection from I into D. It is absurd to the countability of D. □

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